Introduction

The Mathematics Curriculum Guide serves as a guide for teachers when planning instruction and assessments. It defines the content knowledge, skills, and understandings that are measured by the Standards of Learning assessment. It provides additional guidance to teachers as they develop an instructional program appropriate for their students. It also assists teachers in their lesson planning by identifying essential understandings, defining essential content knowledge, and describing the intellectual skills students need to use. This Guide delineates in greater specificity the content that all teachers should teach and all students should learn.

The format of the Curriculum Guide facilitates teacher planning by identifying the key concepts, knowledge, and skills that should be the focus of instruction for each objective. The Curriculum Guide is divided by unit and ordered to match the established HCPS pacing. Each unit is divided into two parts: a one page unit overview and a Teacher Notes and Resource section. The unit overview contains the suggested lessons for the unit and all the DOE curriculum framework information including the related SOL(s), strands, Essential Knowledge and Skills, and Essential Understandings. The Teacher Notes and Resource section is divided by Resources, Key Vocabulary, Essential Questions, Teacher Notes and Elaborations, Honors/AP Extensions, and Sample Instructional Strategies and Activities. The purpose of each section is explained below.

**Vertical Articulation:** This section includes the foundational objectives and the future objectives correlated to each SOL.

**Unit Overview:**
- **Curriculum Information:** This section includes the SOL and SOL Reporting Category, focus or topic, and pacing guidelines.
- **Essential Knowledge and Skills:** Each objective is expanded in this section. What each student should know and be able to do in each objective is outlined. This is not meant to be an exhaustive list nor is a list that limits what taught in the classroom. This section is helpful to teachers when planning classroom assessments as it is a guide to the knowledge and skills that define the objective. (Taken from the Curriculum Framework)
- **Essential Understandings:** This section delineates the key concepts, ideas and mathematical relationships that all students should grasp to demonstrate an understanding of the objectives. (Taken from the Curriculum Framework)

**Teacher Notes and Resources:**
- **Resources:** This section gives textbook resources, links to related Algebra 2 Online! modules, and links to VDOE’s Enhanced Scope and Sequence lessons.
- **Key Vocabulary:** This section includes vocabulary that is key to the objective and many times the first introduction for the student to new concepts and skills.
- **Essential Questions:** This section explains what is meant to be the key knowledge and skills that define the standard.
- **Teacher Notes and Elaborations:** This section includes background information for the teacher. It contains content that is necessary for teaching this objective and may extend the teachers’ knowledge of the objective beyond the current grade level.
- **Extensions:** This section provides content and suggestions to differentiate for honors/Pre-AP level classes.
- **Sample Instructional Strategies and Activities:** This section provides suggestions for varying instructional techniques within the classroom.

Special thanks to Prince William County Public Schools for allowing information from their curriculum documents to be included in this document.
# Algebra II Pacing and Curriculum Guide

## Course Outline

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<th>Third Marking Period at a Glance</th>
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<td><strong>Unit 1</strong>: Absolute Value Equations and Inequalities (AII.4a, AII.6, AII.7)</td>
<td><strong>Unit 4</strong>: Radical Equations and Functions (AII.4d, AII.6, AII.7)</td>
<td><strong>Unit 7</strong>: Systems of Non-Linear Equations (AII.5)</td>
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<td><strong>Unit 3</strong>: Complex Numbers and Simplifying Radicals (AII.3)</td>
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<td><strong>Unit 10</strong>: Exponential and Logarithmic Functions (AII.6, AII.7)</td>
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View the online HCPS Pacing Guide for more details

## Big Ideas

|----------------------------|-------------|-------------------------------|------------------------|--------------------------------------|

## Algebra II SOL Test Blueprint (50 questions total)

<table>
<thead>
<tr>
<th>Expressions and Operations</th>
<th>13 Questions</th>
<th>26% of the Test</th>
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<tbody>
<tr>
<td>Equations and Inequalities</td>
<td>13 Questions</td>
<td>26% of the Test</td>
</tr>
<tr>
<td>Functions and Statistics</td>
<td>24 Questions</td>
<td>48% of the Test</td>
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## Resources

<table>
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<th></th>
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<tr>
<td>SOL Vertical Articulation</td>
</tr>
<tr>
<td>---------------------------</td>
</tr>
<tr>
<td><strong>Algebra 1 Standards</strong></td>
</tr>
<tr>
<td>A1.1 Represent verbal quantitative situations algebraically; evaluate expressions for given replacement values of variables</td>
</tr>
<tr>
<td>Sequences &amp; Operations</td>
</tr>
<tr>
<td>A.2 Perform operations on polynomials - a) apply laws of exponents to perform operations on expressions; b) add/subtract/multiply/divide polynomials; c) factor first and second degree binomials/trinomials (1 or 2 variables)</td>
</tr>
<tr>
<td>Solving and Graphing Equations and Inequalities</td>
</tr>
<tr>
<td>A.4 Solve multistep linear/quadratic equation (in 2 variables) - a) solve literal equation; b) justify steps used in simplifying expressions and solving equations; c) solve quadratic equations (algebraically/graphically); d) solve multistep linear equations (algebraically/graphically); e) solve systems of two linear equations (2 variables algebraically/graphically); f) solve real-world problems involving equations and systems of equations</td>
</tr>
<tr>
<td>A.5 Solve multistep linear inequalities (2 variables) - a) solve multistep linear inequalities (algebraically/graphically); b) justify steps used in solving inequalities; c) solve real-world problems involving inequalities; d) solve systems of inequalities.</td>
</tr>
<tr>
<td>Function Analysis</td>
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<tr>
<td>Function Analysis</td>
</tr>
<tr>
<td>AFDA.1 investigate/analyze function (linear/quadratic exponential/log) families/ characteristics: a) continuity; b) local/abs max/min; c) domain/range; d) zeros; e) intercepts; f) intervals of increasing/decreasing; g) end behaviors; h) asymptotes</td>
</tr>
<tr>
<td>AFDA.4 transfer between/analyze multiple representations of functions (algebraic formulas/ graphs/ tables/words)</td>
</tr>
<tr>
<td>A.2 perform operations on polynomials - c) factor first and second degree binomials/trinomials (1 or 2 variables)</td>
</tr>
<tr>
<td>Data Analysis</td>
</tr>
<tr>
<td>AFDA.3 collect data/generate equation for the curve (linear/quadratic/exponential/log) of best fit/use best fit equations to interpolate function values/make decisions/justify conclusions (algebraic/graph models)</td>
</tr>
<tr>
<td>A.8 given real-world context, analyze relation to determine direct/inverse variation; represent direct variation (algebraically/graphically) and inverse variation (algebraically)</td>
</tr>
<tr>
<td>Variation</td>
</tr>
<tr>
<td>AFDA.7 analyze norm distribution - a) characteristics of normally distribution of data; b) percentiles; c) normalizing data, using ( z )-scores; d) area under standard norm curve/ probability</td>
</tr>
<tr>
<td>AFDA.6 calculate probabilities - a) conditional prob; b) dep/indep events; c) add/mult rules; d) counting techniques (permutations/combinations); Law of Large Numbers</td>
</tr>
<tr>
<td><strong>Strand:</strong> Equations and Inequalities; Expressions and Operations</td>
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<tr>
<td>---------------------------------------------------------------</td>
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</table>

<table>
<thead>
<tr>
<th><strong>SOL AII.3</strong></th>
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<tbody>
<tr>
<td>The student will identify field properties that are valid for the complex numbers.</td>
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<table>
<thead>
<tr>
<th><strong>SOL AII.4a</strong></th>
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</thead>
<tbody>
<tr>
<td>The student will solve, algebraically and graphically, absolute value equations and inequalities. Graphing calculators will be used for solving and for confirming the algebraic solutions.</td>
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<table>
<thead>
<tr>
<th><strong>Essential Knowledge and Skills</strong></th>
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</thead>
<tbody>
<tr>
<td>The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to:</td>
</tr>
<tr>
<td>• Solve absolute value equations and inequalities algebraically and graphically.</td>
</tr>
<tr>
<td>• Apply an appropriate equation to solve a real-world problem.</td>
</tr>
<tr>
<td>• Determine which field properties apply to the complex number system.</td>
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</table>

<table>
<thead>
<tr>
<th><strong>Essential Understandings</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>• The definition of absolute value (for any real numbers (a) and (b), where (b \geq 0), if (</td>
</tr>
<tr>
<td>• Absolute value inequalities can be solved graphically or by using a compound statement.</td>
</tr>
<tr>
<td>• Real-world problems can be interpreted, represented, and solved using equations and inequalities.</td>
</tr>
<tr>
<td>• Equations can be solved in a variety of different ways.</td>
</tr>
<tr>
<td>• Set builder notation may be used to represent solution sets of equations and inequalities.</td>
</tr>
<tr>
<td>• Field properties apply to complex numbers as well as real numbers.</td>
</tr>
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</table>

*(continued)*
### Resources

**Textbook:**
- 1-1 Properties of Real Numbers
- 1-3 Solving Equations
- 1-4 Solving Inequalities
- 1-5 Solving Absolute Value Equations and Inequalities

**HCPS Algebra 2 Online!**:
- Absolute Value Equations

**DOE ESS Lesson Plans:**
- Absolute Value Equations and Inequalities (PDF) (Word)
- Complex Numbers (PDF) (Word)

### Key Vocabulary

- absolute value
- absolute value equation
- absolute value inequality
- compound inequality
- compound sentence
- extraneous solution
- intersection
- union

- field properties –
- associative
- closure
- commutative
- distributive
- identity
- inverse

### Essential Questions

- Why can an absolute value equation take on more than one solution?
- What is a real-world example for an absolute value equation?
- When working with a real-world problem, how are solution(s) verified?
- How is an absolute value equation solved?
- How can the solution for an absolute value inequality be described?

**Teacher Notes and Elaborations**

Absolute value equations and inequalities can be used to model problems dealing with a range of acceptable measurements. Ex: When driving on the interstate, you must have a speed between 40 and 60 miles per hour in the left-hand lane and not get a ticket. Thus your speed, $s$, can be represented by the following equation: $|s - 50| \leq 10$.

When solving *absolute value equations* (equations that contain absolute values), the expression inside the absolute value symbol can be any real number. An absolute value equation may have two solutions.

The absolute value of a number $x$ is the distance the number is from zero on a number line. So if $|2x - 4| = 5$, then the value of $2x - 4$ is 5 units from zero. Thus the value of $2x - 4$ has to be $\pm 5$. To find the solution, you solve the following two equations: $2x - 4 = 5$ and $2x - 4 = -5$.

A solution of an equation makes the equation true for a given value or set of values.

When solving equations it is important to check possible solutions in the original equation as one or more may be an extraneous solution. An *extraneous solution* is a solution of an equation derived from an original equation that is not a solution of the original equation. Absolute value, radical, and rational equations may have extraneous solutions.

The solution of an equation in one variable can be found by graphing... (continued)
Teacher Notes and Elaborations (continued)

Each member of the equation separately and noting the x-coordinate of the point of intersection.

An absolute value inequality may be solved by transforming the inequality into a compound sentence using the words “and” or “or.” In an absolute value inequality, if it is a “less than” statement it is an “and” inequality typically producing one continuous range of solutions. If it is a “greater than” statement it is an “or” inequality typically producing two separate ranges of solutions.

If \(|2x - 4| < 5\), then the value of the expression \(2x - 4\) is less than 5 units from zero on the number line. Thus the value of \(2x - 4\) is between -5 and 5 (‘and’ inequality). So you can solve the inequality \(|2x - 4| < 5\) by solving the compound inequality \(-5 < 2x - 4 < 5\) (\(-5 < 2x - 4\) and \(2x - 4 < 5\)).

If \(|2x - 4| > 5\), then the value of the expression \(2x - 4\) is more than 5 units from zero on the number line. Thus the value of \(2x - 4\) is greater than 5 or less than -5 (‘or’ inequality). You can solve the inequality \(|2x - 4| > 5\) by solving the compound inequality \(2x - 4 > 5\) or \(2x - 4 < -5\).

The sketch of absolute value inequalities and functions presents an opportunity to picture the absolute value relationship. The graphing calculator illustrates various transformations and aids in the solution of absolute value equations and inequalities.

Set builder notation is used to represent solutions. For example, if the solution is \(y = 10\) then in set notation the answer is written \(|y : y = 10\). Or if the solution is \(x < 4\) and \(x > -1\), then in set notation the answer is written as \(|x : -1 < x < 4\).

Graphing calculators are powerful tools for solving and confirming algebraic solutions. Practical problems can be interpreted, represented, and solved using equations.

Sample Instructional Strategies and Activities

- Teachers and students, beginning with natural numbers and progressing to complex numbers, use flow charts and Venn Diagrams as classroom activities.
- Working in cooperative groups, students discover, by point plotting, the "and/or concept" of absolute value. Students, working in cooperative groups of four, write a type of equation (absolute value, quadratic, rational, and radical). Collect and redistribute to different groups. Next, students solve, discuss, and graph the solution. Choose spokesperson to present findings to the class.
- Teach the concept of absolute value, beginning with simplest equation and progressing systematically until complex forms are presented.
- Consider the activity "I have …Who has?" Last card answer will be on the first card. This game can be used for many different types of equations. Each student must work the problems to determine if they have the card with the correct answer.
## Henrico Curriculum Framework
### Algebra 2

### First Nine Weeks

<table>
<thead>
<tr>
<th>Strand: Functions</th>
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<tbody>
<tr>
<td><strong>SOL AII.6</strong> The student will recognize the general shape of absolute value function families and will convert between graphic and symbolic forms of functions. A transformational approach to graphing will be employed. Graphing calculators will be used as a tool to investigate the shapes and behaviors of these functions.</td>
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<table>
<thead>
<tr>
<th>Essential Knowledge and Skills</th>
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<tbody>
<tr>
<td><strong>The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to</strong></td>
</tr>
<tr>
<td>- Recognize graphs of parent functions.</td>
</tr>
<tr>
<td>- Given a transformation of a parent function, identify the graph of the transformed function.</td>
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<tr>
<td>- Given the equation and using a transformational approach, graph a function.</td>
</tr>
<tr>
<td>- Given the graph of a function, identify the parent function.</td>
</tr>
<tr>
<td>- Given the graph of a function, identify the transformations that map the pre-image to the image in order to determine the equation of the image.</td>
</tr>
<tr>
<td>- Using a transformational approach, write the equation of a function given its graph.</td>
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<tr>
<td>- Identify the domain, range, and intercepts of a function presented algebraically or graphically.</td>
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<tr>
<td>- Describe restricted/discontinuous domains and ranges.</td>
</tr>
<tr>
<td>- Given the graph of a function, identify intervals on which the function is increasing and decreasing.</td>
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<tr>
<td>- Describe the end behavior of a function.</td>
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<table>
<thead>
<tr>
<th>Essential Understandings</th>
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<tbody>
<tr>
<td>- The graphs/equations for a family of functions can be determined using a transformational approach.</td>
</tr>
<tr>
<td>- Transformations of graphs include translations, reflections, and dilations.</td>
</tr>
<tr>
<td>- A parent graph is an anchor graph from which other graphs are derived with transformations.</td>
</tr>
<tr>
<td>- Functions may be used to model real-world situations.</td>
</tr>
<tr>
<td>- The domain and range of a function may be restricted algebraically or by the real-world situation modeled by the function.</td>
</tr>
<tr>
<td>- A function can be described on an interval as increasing, decreasing, or constant.</td>
</tr>
<tr>
<td>- End behavior describes a function as $x$ approaches positive and negative infinity.</td>
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</table>

*Characteristics of Functions appear in several units. When discussing this topic include the following items: domain, range, intercepts, intervals over which the function is increasing/decreasing; and end behavior.

(continued)
Absolute Value Functions (continued)

- What is the relationship between domain and range?
- How are the x- and y-intercepts determined?
- What is meant by the end behavior of a function?
- Based on the equation of a function, how do you distinguish between a horizontal translation and a vertical translation?

Teacher Notes and Elaborations

*Absolute value function* is a function described by \( f(x) = |x| \). The graph of the absolute value function has a characteristic V-shape.

- \( f(x) = a|x-h| + k \)
- \( a \) is positive
- \( a \) is negative
- Vertex
  - \( (h, k) \)
  - \( (h, k) \)
- Axis of Symmetry
  - \( x = h \)
  - \( x = h \)
- Direction of Opening
  - upward (minima)
  - downward (maxima)
- As the value of the absolute value of \( a \) increases, the graph of \( f(x) = a|x-h| + k \) narrows.

A function is a correspondence in which values of one variable determine the values of another. It is a rule of correspondence between two sets such that there is a unique element in one set assigned to each element in the other.

(continued)
Absolute Value Functions (continued)

A graph’s shape is determined by the rule or relation which defines it. Very often multiplication or division is involved. A line’s slope and a parabola’s stretch or compression is attributed to dilation – expansion, growing or shrinking, multiplication. If \( y = f(x) \), then \( y = af(x) \) gives a vertical stretch or vertical shrink of the graph of \( f \). If \( a > 1 \), the graph is stretched vertically by a factor of \( a \). If \( 0 < a < 1 \), the graph is compressed vertically by a factor of \( a \).

Rules recap:
\[
\begin{align*}
  f(x) + a & \text{ is } f(x) \text{ shifted upward } a \text{ units} \\
  f(x) - a & \text{ is } f(x) \text{ shifted downward } a \text{ units} \\
  f(x + a) & \text{ is } f(x) \text{ shifted left } a \text{ units} \\
  f(x - a) & \text{ is } f(x) \text{ shifted right } a \text{ units} \\
  -f(x) & \text{ is } f(x) \text{ reflected over the } x\text{-axis} \\
  f(-x) & \text{ is } f(x) \text{ reflected over the } y\text{-axis} \\
  af(x) & \text{ is } f(x) \text{ with a vertical stretch or shrink}
\end{align*}
\]

Functions describe the relationship between two variables. A function is continuous if the graph can be drawn without lifting the pencil from the paper. A graph is discontinuous if it has jumps, breaks, or holes in it. Each function, whether continuous or discontinuous, has a distinct domain, range, zero(s), \( y \)-intercept, and inverse.

The domain is the set of all possible values for the first coordinate of a function. The range of a function is the set of all possible values for the second coordinate of a function.

The domain of every absolute value function is the set of all real numbers. As a result, the graph of an absolute value function extends infinitely. What happens to an absolute value function as its domain values get very small and very large is called the end behavior of an absolute value function.

### Teacher Notes and Elaborations (continued)

The graphs and/or equations for a family of functions can be determined using a transformational approach. A family of functions is a group of functions with common characteristics. A parent function is the simplest function with these characteristics. A parent function and one or more transformations make up a family of functions. Shapes and behavior of graphs of polynomials can be determined by analyzing transformations of parent functions.

The following is one example of a parent function and family of functions.

<table>
<thead>
<tr>
<th>Parent Function</th>
<th>Family of Functions</th>
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<tbody>
<tr>
<td>( f(x) =</td>
<td>x</td>
</tr>
<tr>
<td>( f(x) = -2</td>
<td>x+1</td>
</tr>
</tbody>
</table>

A transformation of a function is an alteration of the function rule that results in an alteration of its graph.

If \( y = f(x) \), then \( y = f(x) + k \) gives a vertical translation of the graph of \( f \). The translation is \( k \) units up for \( k > 0 \) and \( |k| \) units down for \( k < 0 \). If \( y = f(x) \), then \( y = f(x - h) \) gives a horizontal translation of the graph of \( f \). The translation is \( h \) units to the right for \( h > 0 \) and \( |h| \) units to the left for \( h < 0 \). Note: the standard form for a horizontal translation contains a subtraction sign. Thus when determining the direction of the shift, students need to separate the value of \( h \) from the operation seen in front of it (it is the ‘opposite’ of the obvious value). If the function is given as \( y = f(x + h) \), then the value of \( h \) is actually negative (\( y = f(x - (-h)) \)).

If \( y = f(x) \), then \( y = -f(x) \) gives a reflection of the graph of \( f \) across the \( x\)-axis. If \( y = f(x) \), then \( y = f(-x) \) gives a reflection of the graph of \( f \) across the \( y\)-axis.

Return to Course Outline
### Teacher Notes and Elaborations (continued)

The **x-intercept** is the x-coordinate of the point where the graph crosses the x-axis. The **y-intercept** is the y-coordinate of the point where the graph crosses the y-axis.

A function is *increasing* on an interval if its graph always rises as it moves from left to right over the interval. It is *decreasing* on an interval if its graph always falls as it moves from left to right over the interval.

A function is constant on an interval if its graph is horizontal over the interval. For any $x_1$ and $x_2$ in the interval, where $x_1 < x_2$,

then $f(x_1) = f(x_2)$

For a function $y = f(x)$:

- When $x_1 < x_2$, then $f(x_1) \leq f(x_2)$ the function is increasing.
- When $x_1 > x_2$, then $f(x_1) \geq f(x_2)$ the function is decreasing.

Example:

![Graph of an absolute value function](image)

This function is constant on the interval $[-5, 2]$, decreasing on the interval $(2, 3)$, and increasing on the interval $[3, \infty)$.

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### Honors/Pre-AP Extension

If $y = f(x)$, then $y = f(bx)$, gives a horizontal stretch or horizontal compression of the graph of $f$. If $b > 1$, the graph is compressed horizontally by a factor of $\frac{1}{b}$. If $0 < b < 1$, the graph is stretched horizontally by a factor of $\frac{1}{b}$.

---

### Sample Instructional Strategies and Activities

- The students are divided into small groups and given graph representations of the following functions: **absolute value**, square root, cube root, rational, polynomial, exponential, and logarithmic. They must identify the graph representations and explain the transformations from the basic graph of each function. Each group must verify their answers on the graphing calculator.

- Students, working in small groups, will be given several selective absolute value functions. They will use the graphing calculator to graph these functions. Next, each group will draw conclusions about the general shape, end behavior, zeros, and y-intercept of the functions.
<table>
<thead>
<tr>
<th><strong>Strand</strong>: Expressions and Operations; Equations and Inequalities</th>
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</table>

**SOL AII.1bc**
The student, given radical expressions, will
b) add, subtract, multiply, divide, and simplify radical expressions containing rational numbers and variables, and expressions containing rational exponents;
c) write radical expressions as expressions containing rational exponents and vice versa.

**Essential Knowledge and Skills**
The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to
- Simplify radical expressions containing positive rational numbers and variables.
- Convert from radical notation to exponential notation, and vice versa.
- Add and subtract radical expressions.
- Multiply and divide radical expressions not requiring rationalizing the denominators.

**Essential Understandings**
- Radical expressions can be written and simplified using rational exponents.
- Only radicals with a common radicand and index can be added or subtracted.

(continued)
Radical Expressions and Rational Exponents (continued)

The number 5 under the radical sign is called the radicand. In \( \sqrt{7} \), 3 is called the index.

Radicals with common radicands and indices are added and subtracted the same way monomials are added and subtracted. The coefficient changes but the radical stays the same. To multiply radicals, only the indices must be the same. When multiplying radicals, the product of the coefficients is the new coefficient, and the product of the radicands becomes the new radicand.

There are two popular ways to simplify radicals:

- **Product Property of Radicals** – For any real numbers \( a \) and \( b \) and any integer \( n, n > 1 \), then \( \sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \) where \( a \) and \( b \) are both nonnegative, and if \( n \) is odd, then \( \sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \).

  Rewrite the radicand as the product of two factors, one of which is a ‘perfect’ square/cube/etc. (ideally the largest perfect factor).

  \[
  \sqrt{72} = \sqrt{36 \cdot 2} = \sqrt{36} \cdot \sqrt{2} = 6\sqrt{2} \\
  \sqrt{72} = \sqrt{8 \cdot 9} = \sqrt{8} \cdot \sqrt{9} = 2\sqrt{9}
  \]

- **Quotient Property of Radicals** – For any real numbers \( a \) and \( b \), where \( b \neq 0 \), and for any integer \( n, n > 1 \), \( \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \), if all roots are defined.
### Teacher Notes and Elaborations (continued)

Expressions containing radicals can be operated on and simplified. A radical expression is simplified when the following conditions are met:

- the index, \( n \), has the least value possible;
- the radicand contains no factors (other than 1) that are \( n \)th powers of an integer or polynomial;
- the radicand contains no fractions; and
- no radicals appear in the denominator.

Rationalizing a denominator is a procedure for transforming a quotient with a radical in the denominator into an expression with no radical in the denominator. The following are examples of rationalizing the denominator of radical expressions.

**Example 1:**

\[
\frac{x}{\sqrt[3]{x}} = \frac{x}{\sqrt[3]{x}} \cdot \left(\frac{\sqrt[3]{x}}{\sqrt[3]{x}}\right) = \frac{x \cdot \sqrt[3]{x}}{\sqrt[3]{x} \cdot \sqrt[3]{x}} = \frac{x\sqrt[3]{x}}{3}
\]

**Example 2:**

\[
\frac{2}{1+\sqrt{3}} = \frac{2}{1+\sqrt{3}} \cdot \left(\frac{1-\sqrt{3}}{1-\sqrt{3}}\right) = \frac{2-2\sqrt{3}}{1-\sqrt{3} + \sqrt{3} - 3} = \frac{2-2\sqrt{3}}{-2} = -1 + \sqrt{3}
\]

**Example 3:**

\[
\sqrt{\frac{2}{3x}} = \frac{\sqrt[3]{2}}{\sqrt[3]{3x}} \cdot \left(\frac{\sqrt[3]{3^2 x^2}}{\sqrt[3]{3^2 x^2}}\right) = \frac{\sqrt[3]{2 \cdot 3^2 x^2}}{\sqrt[3]{3^2 x^2}} = \frac{\sqrt[3]{18x^2}}{3x}
\]

Students need multiple experiences simplifying radical expressions with and without radicals in the denominator.

### Sample Instructional Strategies and Activities

- **Model problems containing roots such as ratios of areas and volumes.**
- **Divide the class into groups of four students. Give each group four problems that involve radicals. Each of the four operations should be used. Have each group explain how to do the problems step by step. The groups should make sure that every person in the group understands each problem before presenting them.**

---

**A rational exponent** is an exponent written in fraction form. Rational exponents provide an alternative way to express radicals and allow for radicals with different indices to be multiplied using the rules of exponents. For any nonzero real number \( b \), and any integers \( m \) and \( n \), with \( n > 1 \),

\[
b^\frac{m}{n} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m
\]

(except when \( b < 0 \) and \( n \) is even).

\[
\sqrt[2]{8^3} = \frac{3}{2} \times \frac{8^2}{2} = \left(\sqrt[2]{8}\right)^2 = (2)^2 = 4
\]

\[
\sqrt{10x^2 y} \cdot \sqrt{x^2 y^2} = 10^2 x^2 y^2 \cdot x^3 y^3 = 10^6 x^6 y^6 \cdot x^4 y^6 =
\]

\[
10^6 x^6 y^6 = \sqrt[1000]{10^7} = xy \sqrt[1000]{x^4 y}
\]
### Complex Numbers

**Strand:** Expressions and Operations

**SOL AII.3**
The student will perform operations on complex numbers, express the results in simplest form using patterns of the powers of $i$, and identify field properties that are valid for the complex numbers.

**SOL AII.4bd**
The student will solve, algebraically and graphically,

b) quadratic equations over the set of complex numbers;

d) equations containing radical expressions.

Graphing calculators will be used for solving and for confirming the algebraic solutions.

---

**Essential Knowledge and Skills**

The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to

- Recognize that the square root of $-1$ is represented as $i$.
- Simplify radical expressions containing negative rational numbers and express in $a + bi$ form.
- Simplify powers of $i$.
- Add, subtract, and multiply complex numbers.
- Write a real number in $a + bi$ form.
- Write a pure imaginary number in $a + bi$ form.
- Place the following sets of numbers in a hierarchy of subsets: complex, pure imaginary, real, rational, irrational, integers, whole, and natural.
- Determine which field properties apply to the complex number system.
- Solve an equation containing a radical expression algebraically and graphically.
- Solve a quadratic equation over the set of complex numbers using an appropriate strategy.
- Verify possible solutions to an equation containing radical expressions.
- Apply an appropriate equation to solve a real-world problem.

---

**Essential Understandings**

- Complex numbers are organized into a hierarchy of subsets.
- Field properties apply to complex numbers as well as real numbers.
- A complex number multiplied by its conjugate is a real number.
- All complex numbers can be written in the form $a + bi$ where $a$ and $b$ are real numbers and $i$ is $\sqrt{-1}$.
- Equations having no real number solutions may have solutions in the set of complex numbers.
- The process of solving radical equations can lead to extraneous solutions.
- Equations can be solved in a variety of ways.
- Set builder notation may be used to represent solution sets of equations and inequalities.
- Real-world problems can be interpreted, represented, and solved using equations and inequalities.

(continued)
Complex Numbers (continued)

Complex numbers are a superset of real numbers and, as a system, contain solutions for equations that are not solvable over the set of real numbers.

Complex numbers are organized into a hierarchy of subsets with properties applicable to each subset.

A complex number (the sum of a real number and an imaginary number) can be written in the form $a + bi$ where $a$ and $b$ are real numbers and $i = \sqrt{-1}$. An imaginary number (the square root of a negative number) is a complex number $(a + bi)$ where $b \neq 0$, and $i = \sqrt{-1}$. An imaginary number where $a = 0$ is called a pure imaginary number.

The set of complex numbers form a field (Properties of closure, associative, commutative, inverse, and identity of addition and multiplication with the distributive property are valid.).

Operations of addition, subtraction, and multiplication of complex numbers are performed in the same manner as the respective operations for radicals. The conjugate (expressions that differ only in the sign of the second term) of $a + bi$ is $a - bi$. The product of two complex conjugates is a real number. When dividing complex numbers the conjugate of the denominator must be used to simplify the expression.
Teacher Notes and Elaborations (continued)

Complex Numbers (continued)

Real numbers are not adequate to determine the solutions of an equation such as $x^2 + 1 = 0$, because there is no real number $x$ that can be squared to produce $-1$. With the definition of $i^2 = -1$, and $i = \sqrt{-1}$, the sets of imaginary and complex numbers are created. The standard form of any complex number is $a + bi$ with $a, b \in \mathbb{R}$.

Operations of addition, subtraction, and multiplication are performed in the same manner as the respective operations for radicals. All answers must be simplified to standard form using the pattern of the powers of $i$ ($i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$).

A radical expression is an expression that contains a square root. Equations with variables in the radicals are called radical equations. To solve this type of equation, isolate the radical and then raise each side of the equation to a power that equals the root of the equation. For example, if the root index is 3, then each side of the equation needs to be raised to the $3^{rd}$ power. If the root index is 4, then each side is raised to the $4^{th}$ power. During this unit, raising both sides to a power should not create a quadratic equation.

A solution of an equation makes the equation true for a given value or set of values. An extraneous solution is a solution of an equation derived from an original equation that is not a solution of the original equation. Absolute value, radical, and rational equations may have extraneous solutions.

Polynomial equations can be solved by using radicals if there is a single variable term (extracting roots). Isolate the variable term and, matching the root to the index, take the root of both sides.

$$2x^2 - 5 = -37 \quad \rightarrow \quad 2x^2 = -32 \quad \rightarrow \quad x^2 = -16$$

$$\sqrt{x^2} = \sqrt{-16} \quad \rightarrow \quad x = 4i$$

Set builder notation is used to represent solutions. For example, if the solution is $y = 10$ then in set notation the answer is written \{$y : y = 10$\}.

Graphing calculators are powerful tools for solving and confirming algebraic solutions. Practical problems can be interpreted, represented, and solved using equations.

Sample Instructional Strategies and Activities

- Make a set of about 30 cards, one per card, $i$ to a certain power. Pair the students and have them write on a white board the simplified result. After completing the cards, students should be able to generalize how to simplify $i$ raised to a power.
- Teachers and students, beginning with natural numbers and progressing to complex numbers, use flow charts and Venn Diagrams, as classroom activities.
- Students, working in cooperative groups of four, write a type of equation (absolute value, quadratic, rational, and radical). Collect and redistribute to different groups. Next, students solve, discuss, and graph the solution. Choose spokesperson to present findings to the class.
- Consider the activity "I have …Who has?" Last card answer will be on the first card. This game can be used for many different types of equations. Each student must work the problems to determine if they have the card with the correct answer.
- Working in cooperative groups, ask students to discuss an appropriate method for solving equations involving two radicals. Students will verbalize each step needed and discuss the proper steps needed.
### Strand: Functions

SOL AII.6 The student will recognize the general shape of function (square root, cube root, and polynomial) families and will convert between graphic and symbolic forms of functions. A transformational approach to graphing will be employed.

SOL AII.7 The student will investigate and analyze functions algebraically and graphically. Key concepts include:
- Domain and range, including limited and discontinuous domains and ranges;
- Zeros;
- \( x \)- and \( y \)-intercepts;
- Intervals in which a function is increasing or decreasing;
- End behavior;
Graphing calculators will be used as a tool to investigate the shapes and behaviors of these functions.

### Essential Knowledge and Skills

The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to:

- Recognize graphs of parent functions.
- Given a transformation of a parent function, identify the graph of the transformed function.
- Given the equation and using a transformational approach, graph a function.
- Given the graph of a function, identify the parent function.
- Given the graph of a function, identify the transformations that map the pre-image to the image in order to determine the equation of the image.
- Using a transformational approach, write the equation of a function given its graph.
- Identify the domain, range, zeros, and intercepts of a function presented algebraically or graphically.
- Describe restricted/discontinuous domains and ranges.
- Given the graph of a function, identify intervals on which the function is increasing and decreasing.
- Describe the end behavior of a function.

### Essential Understandings

- The graphs/equations for a family of functions can be determined using a transformational approach.
- Transformations of graphs include translations, reflections, and dilations.
- A parent graph is an anchor graph from which other graphs are derived with transformations.
- Functions may be used to model real-world situations.
- The domain and range of a function may be restricted algebraically or by the real-world situation modeled by the function.
- A function can be described on an interval as increasing, decreasing, or constant.
- End behavior describes a function as \( x \) approaches positive and negative infinity.
- A zero of a function is a value of \( x \) that makes \( f(x) \) equal zero.
### Radical Functions (continued)

#### Resources

**Textbook:**
- 2-6 Vertical and Horizontal Translations
- pg. Geometric Transformation Extension
- pg. 306 End Behavior Extension
- 7-8 Graphing Radical Functions

**HCPS Algebra 2 Online!:**
- Function Families
- Characteristics of Functions

**DOE ESS Lesson Plans:**
- Transformational Graphing (PDF) (Word)
- Functions: Domain, Range, End Behavior, Increasing, Decreasing (PDF) (Word)
- Composition of Functions (PDF) (Word)
- Inverse Functions (PDF) (Word)

### Essential Questions
- What is the transformational approach to graphing?
- What is the connection between the algebraic and graphical representation of a transformation?
- What is a function?
- What is the relationship between domain and range?
- What is the relationship between a function and its inverse?
- What operations can be performed on functions?
- What is the relationship between the degree of a function, the graph of a function, and the number of zeros of a function?
- How can the calculator be used to investigate the shape and behavior of polynomial functions?
- How are the x- and y-intercepts determined?
- What is meant by the end behavior of a function?
- Describe the characteristics of the graphs of odd-degree and even-degree polynomial functions whose leading coefficients are positive. How does negative leading coefficients change each of these?
- What is meant by the turning points of a function and how are they found?

### Key Vocabulary

- continuous
- cube root function
- cubic function
- decreasing
- dilation
- discontinuous
- domain
- end behavior
- family of functions
- increasing
- intercepts (x and y)
- multiplicities
- parent function
- polynomial function
- quartic function
- range
- reflection
- repeating zeros
- square root function
- transformations of graphs
- translation (vertical/horizontal)

### Teacher Notes and Elaborations

A function is a correspondence in which values of one variable determine the values of another. It is a rule of correspondence between two sets such that there is a unique element in one set assigned to each element in the other.

Radical functions also produce families of functions. The following radical functions should be included:
Radical Functions (continued)

Teacher Notes and Elaborations (continued)

<table>
<thead>
<tr>
<th>Square Root</th>
<th>Cubed Root</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \sqrt{x} )</td>
<td>( f(x) = \sqrt[3]{x} )</td>
</tr>
</tbody>
</table>

A transformation of a function is an alteration of the function rule that results in an alteration of its graph.

If \( y = f(x) \), then \( y = f(x) + k \) gives a **vertical translation** of the graph of \( f \). The translation is \( k \) units up for \( k > 0 \) and \( |k| \) units down for \( k < 0 \). If \( y = f(x) \), then \( y = f(x - h) \) gives a **horizontal translation** of the graph of \( f \). The translation is \( h \) units to the right for \( h > 0 \) and \( |h| \) units to the left for \( h < 0 \). Note: the standard form for a horizontal translation contains a subtraction sign. Thus when determining the direction of the shift, students need to separate the value of \( h \) from the operation seen in front of it (it is the ‘opposite’ of the obvious value). If the function is given as \( y = f(x + h) \), then the value of \( h \) is actually negative (\( y = f(x - (-h)) \)).

If \( y = f(x) \), then \( y = -f(x) \) gives a **reflection** of the graph of \( f \) across the \( x \)-axis. If \( y = f(x) \), then \( y = f(-x) \) gives a reflection of the graph of \( f \) across the \( y \)-axis.

A curve’s shape is determined by the rule or relation which defines it. Very often multiplication or division is involved. A line’s slope and a parabola’s stretch or compression is attributed to **dilation** – expansion, growing or shrinking, multiplication. If \( y = f(x) \), then \( y = af(x) \) gives a vertical stretch or vertical compression of the graph of \( f \). If \( a > 1 \), the graph is stretched vertically by a factor of \( a \). If \( 0 < a < 1 \), the graph is compressed vertically by a factor of \( a \).

Rules recap:

- \( f(x) + a \) is \( f(x) \) shifted upward \( a \) units
- \( f(x) - a \) is \( f(x) \) shifted downward \( a \) units
- \( f(x + a) \) is \( f(x) \) shifted left \( a \) units
- \( f(x - a) \) is \( f(x) \) shifted right \( a \) units
- \( -f(x) \) is \( f(x) \) reflected over the \( x \)-axis
- \( f(-x) \) is \( f(x) \) reflected over the \( y \)-axis
- \( af(x) \) is \( f(x) \) with a vertical stretch or shrink

Functions describe the relationship between two variables. A function is **continuous** if the graph can be drawn without lifting the pencil from the paper. A graph is **discontinuous** if it has jumps, breaks, or holes in it. Each function, whether continuous or discontinuous, has a distinct domain, range, zero(s), \( y \)-intercept, and inverse.

The **domain** is the set of all possible values for the first coordinate of a function. The **range** of a function is the set of all possible values for the second coordinate of a function.

The **\( x \)-intercept** is the \( x \)-coordinate of the point where the graph crosses the \( x \)-axis and has the ordered pair \((x, 0)\). The **\( y \)-intercept** is the \( y \)-coordinate of the point where the graph crosses the \( y \)-axis and has the ordered pair \((0, y)\).

A function is **increasing** on an interval if its graph always rises as it... (continued)
Teacher Notes and Elaborations (continued)

moves from left to right over the interval. It is decreasing on an interval if its graph always falls as it moves from left to right over the interval.

For a function \( y = f(x) \):
- When \( x_1 < x_2 \), then \( f(x_1) \leq f(x_2) \) the function is increasing.
- When \( x_1 > x_2 \), then \( f(x_1) \geq f(x_2) \) the function is decreasing.

A function is constant on an interval if its graph is horizontal over the interval. For any \( x_1 \) and \( x_2 \) in the interval, where \( x_1 < x_2 \), then \( f(x_1) = f(x_2) \)

Example:

This function is constant on the interval \([-5, 2]\), decreasing on the interval \((2, 3)\), and increasing on the interval \([3, \infty)\).

Honors/Pre-AP Extension

If \( y = f(x) \), then \( y = f(bx) \), gives a horizontal stretch or horizontal compression of the graph of \( f \). If \( b > 1 \), the graph is compressed horizontally by a factor of \( \frac{1}{b} \). If \( 0 < b < 1 \), the graph is stretched horizontally by a factor of \( \frac{1}{b} \).

Sample Instructional Strategies and Activities

- The students are divided into small groups and given graph representations of the following functions: absolute value, square root, cube root, rational, polynomial, exponential, and logarithmic. They must identify the graph representations and explain the transformations from the basic graph of each function. Each group must verify their answers on the graphing calculator.
- Students, working in small groups, will be given several selective polynomial functions. They will use the graphing calculator to categorize these functions. Next, each group will draw conclusions about the general shape, end behavior, zeros, and \( y \)-intercept of the functions.
- On cards write 20 functions beginning with \( a(x), b(x), c(x) \) …. Mix the cards and select two at random. Have students write the composition of the two functions [e.g., \( a(d(x)) \)]. Using the same functions have students reverse the order [e.g., \( (d(a(x))) \)]. Is composition of functions commutative? Use the same cards and have students perform the four operations.
### Factoring Polynomials

**Strand:** Expressions and Operations

**SOL AII.1d**
The student, given polynomial expressions, will factor polynomials completely.

<table>
<thead>
<tr>
<th>Essential Knowledge and Skills</th>
</tr>
</thead>
<tbody>
<tr>
<td>The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to</td>
</tr>
<tr>
<td>• Factor polynomials by applying general patterns including difference of squares, sum and difference of cubes, and perfect square trinomials.</td>
</tr>
<tr>
<td>• Factor polynomials completely over the integers.</td>
</tr>
<tr>
<td>• Verify polynomial identities including the difference of squares, sum and difference of cubes, and perfect square trinomials.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Essential Understandings</th>
</tr>
</thead>
<tbody>
<tr>
<td>• The complete factorization of polynomials has occurred when each factor is a prime polynomial.</td>
</tr>
<tr>
<td>• Pattern recognition can be used to determine complete factorization of a polynomial.</td>
</tr>
</tbody>
</table>

(continued)
Teacher Notes and Resources

**Resources**

**Textbook:**
- 6-2 Polynomials and Linear Factors (example 1)
- pg. 853 Factoring & Operations with Polynomials
- 5-4 Factoring Quadratic Expressions
- 6-4 Solving Polynomial Equations

**HCPS Algebra 2 Online!**:
- Factoring Polynomials

**DOE ESS Lesson Plans:**
- Factoring – Expressions and Operations (PDF) (Word)

---

**Key Vocabulary**

difference of two cubes  
difference of squares  
factoring  
perfect square trinomial  
polynomial  
sum of two cubes

---

**Essential Questions**

- When is a polynomial completely factored?
- What are the patterns to investigate when factoring a polynomial?

---

**Teacher Notes and Elaborations**

The following steps may be followed when factoring a polynomial:

1. Determine the greatest monomial factor (GCF) as a first step in complete factorization.
2. Check for special patterns
   - Difference of squares [e.g., \((4x^2 - 25) = (2x - 5)(2x + 5)\)]
   - Sum of two cubes [e.g., \((x^3 + 8) = (x + 2)(x^2 - 2x + 4)\)]
   - Difference of two cubes [e.g., \((y^3 - 125) = (y - 5)(y^2 + 5y + 25)\)]
   - Perfect square trinomials [e.g., \((y^2 - 12y + 36) = (y - 6)^2\)]

3. If there are four or more terms, grouping should be tried.

To check, the factors may be multiplied back together to see if the original polynomial results.

An equation that is true for all real numbers for which both sides are defined is called an identity. A polynomial identity is two equivalent polynomial expressions. To verify or prove polynomial identities, work with the expressions on each side of the equation until they are the same. The following are examples of proving polynomial identities.

*Problems such as those below often lend themselves to discussions about the properties, especially distributive, commutative, and associative, and additive inverse.

**Example 1:** Prove that \(x^2 - y^2 = (x - y)(x + y)\)

\[
= x^2 + xy - xy - y^2 \\
= x^2 - y^2
\]

**Example 2:** Prove that \((a + b)(a^2 - ab + b^2) = a^3 + b^3\)

\[
(a + b)(a^2 - ab + b^2) = a^3 + b^3 \\
a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3 = \\
a^3 - a^2b + a^2b + ab^2 - ab^2 + b^3 = \\
a^3 + b^3
\]

**Example 3:** Prove that \((a + b)^2 + (a - b)^2 = 2a^2 + 2b^2\)

\[
a^2 + 2ab + b^2 + a^2 - 2ab + b^2 = \\
a^2 + a^2 + 2ab - 2ab + b^2 + b^2 = \\
2a^2 + 2b^2
\]
Teacher Notes and Elaborations (continued)

Example 4: Use polynomial identities to describe numerical relationships such as \((x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2\) to generate Pythagorean triples.

\[
(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2
\]
\[
x^4 + x^2y^2 + x^2y^2 + y^4 = x^4 - x^2y^2 - x^2y^2 + y^4 + 4x^2y^2
\]
\[
x^4 + 2x^2y^2 + y^4 = x^4 - 2x^2y^2 + y^4 + 4x^2y^2
\]
\[
x^4 + 2x^2y^2 + y^4 = x^4 + 4x^2y^2 - 2x^2y^2 + y^4
\]
\[
x^4 + 2x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4
\]

Sample Instructional Strategies and Activities

- Consider the activity "I have... Who has...?"

A sample card might say:

<table>
<thead>
<tr>
<th>I have ((x - 2)(x + 2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Who has the factors of (6x^2 - 7x - 5)</td>
</tr>
</tbody>
</table>

The person having a card that says "I have \((2x + 1)(3x - 5)\)" would then read their card. When making a set of cards, be sure that the answer to the question on the last card answer will be on the first card. This game can be used for many different types of problems (i.e., factoring polynomials; and adding, subtracting, multiplying, and dividing integers, rational numbers, complex numbers, etc.). Each student must work the problems to determine if they have the card with the correct answer.

Return to Course Outline
## Second Nine Weeks

### Solving Quadratic Equations & Making Connections

**Strand**: Equations and Inequalities; Functions

**SOL AII.4b**
The student will solve, algebraically and graphically, quadratic equations over the set of complex numbers. Graphing calculators will be used for solving and for confirming the algebraic solutions.

**SOL AII.8**
The student will investigate and describe the relationships among solutions of an equation, zeros of a function, \( x \)-intercepts of a graph, and factors of a polynomial expression.

### Essential Knowledge and Skills

The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to:

- Solve a quadratic equation over the set of complex numbers using an appropriate strategy.
- Calculate the discriminant of a quadratic equation to determine the number of real and complex solutions.
- Apply an appropriate equation to solve a real-world problem.
- Recognize that the quadratic formula can be derived by applying the completion of squares to any quadratic equation in standard form.
- Describe the relationships among solutions of an equation, zeros of a function, \( x \)-intercepts of a graph, and factors of a polynomial expression.
- Define a polynomial function, given its zeros.
- Determine a factored form of a polynomial expression from the \( x \)-intercepts of the graph of its corresponding function.
- For a function, identify zeros of multiplicity greater than 1 and describe the effect of those zeros on the graph of the function.
- Given a polynomial equation, determine the number of real solutions and non-real solutions.

### Essential Understandings

- A quadratic function whose graph does not intersect the \( x \)-axis has roots with imaginary components.
- The quadratic formula can be used to solve any quadratic equation.
- The value of the discriminant of a quadratic equation can be used to describe the number of real and complex solutions.
- Real-world problems can be interpreted, represented, and solved using equations and inequalities.
- Equations can be solved in a variety of ways.
- Set builder notation may be used to represent solution sets of equations and inequalities.
- The *Fundamental Theorem of Algebra* states that, including complex and repeated solutions, an \( n \)th degree polynomial equation has exactly \( n \) roots (solutions).
- The following statements are equivalent:
  - \( k \) is a zero of the polynomial function \( f \);
  - \( (x - k) \) is a factor of \( f(x) \);
  - \( k \) is a solution of the polynomial equation \( f(x) = 0 \); and \( k \) is an \( x \)-intercept for the graph of \( y = f(x) \).

(continued)
Solving Quadratic Equations & Making Connections (continued)

Teacher Notes and Elaborations

A quadratic equation is a polynomial equation containing a variable to the second degree. The general form of a quadratic equation is $y = ax^2 + bx + c$, where $a$, $b$, and $c$ are real numbers and $a \neq 0$. The graph of a quadratic function is called a parabola. Quadratic functions can be written in the standard form $y = a(x - h)^2 + k$ to facilitate graphing by transformations and finding maxima and minima.

The graphs and/or equations for a family of functions can be determined using a transformational approach. A family of functions is a group of functions with common characteristics. A parent function is the simplest function with these characteristics. A parent function and one or more transformations make up a family of functions. Shapes and behavior of graphs of polynomials can be determined by analyzing transformations of parent functions.

The following example is of the quadratic parent function and family of functions.

<table>
<thead>
<tr>
<th>Parent Function</th>
<th>Family of Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = x^2$</td>
<td>$f(x) = 3(x - 4)^2 + 5$</td>
</tr>
<tr>
<td></td>
<td>$f(x) = -2(x + 1)^2 - 7$</td>
</tr>
</tbody>
</table>

(continued)
A quadratic equation is a polynomial equation containing a variable to the second degree. The graph of a quadratic function is called a parabola. Parabolas have an axis of symmetry, a line that divides the parabola into two parts that are mirror images of each other. The vertex of a parabola is either the lowest point on the graph or the highest point on the graph.

The chart below summarizes the characteristics of the graph of $y = a(x-h)^2 + k$.

<table>
<thead>
<tr>
<th>$y = a(x-h)^2 + k$</th>
<th>$a$ is positive</th>
<th>$a$ is negative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertex</td>
<td>$(h, k)$</td>
<td>$(h, k)$</td>
</tr>
<tr>
<td>Axis of Symmetry</td>
<td>$x = h$</td>
<td>$x = h$</td>
</tr>
<tr>
<td>Direction of Opening</td>
<td>upward (minima)</td>
<td>downward (maxima)</td>
</tr>
</tbody>
</table>

As the value of the absolute value of $a$ increases, the graph of $y = a(x-h)^2 + k$ narrows.

A u-turn in a graph occurs when the function values change from increasing to decreasing or vice versa and indicates either a local maximum or a local minimum. The number of u-turns in a graph is no greater than 1 less than the degree of the function. For a quadratic function (degree 2) there is one u-turn.

A transformation of a function is an alteration of the function rule that results in an alteration of its graph.

If $y = f(x)$, then $y = f(x) + k$ gives a vertical translation of the graph of $f$. The translation is $k$ units up for $k > 0$ and $|k|$ units down for $k < 0$. If $y = f(x)$, then $y = f(x-h)$ gives a horizontal translation of the graph of $f$. The translation is $h$ units to the right for $h > 0$ and $|h|$ units to the left for $h < 0$. Note: the standard form for a horizontal translation contains a subtraction sign. Thus when determining the direction of the shift, students need to separate the value of $h$ from the operation seen in front of it (it is the ‘opposite’ of the obvious value). If the function is given as $y = f(x + h)$, then the value of $h$ is actually negative ($y = f(x - (-h))$).

If $y = f(x)$, then $y = -f(x)$ gives a reflection of the graph of $f$ across the x-axis. If $y = f(x)$, then $y = f(-x)$ gives a reflection of the graph of $f$ across the y-axis.

A curve’s shape is determined by the rule or relation which defines it. Very often multiplication or division is involved. A line’s slope and a parabola’s stretch or compression is attributed to dilation – expansion, growing or shrinking, multiplication. If $y = f(x)$, then $y = af(x)$ gives a vertical stretch or vertical compression of the graph of $f$. If $a > 1$, the graph is stretched vertically by a factor of $a$. If $0 < a < 1$, the graph is compressed vertically by a factor of $a$.

Rules recap:

- $f(x) + a$ is $f(x)$ shifted upward $a$ units
- $f(x) - a$ is $f(x)$ shifted downward $a$ units
- $f(x + a)$ is $f(x)$ shifted left $a$ units
- $f(x - a)$ is $f(x)$ shifted right $a$ units
- $-f(x)$ is $f(x)$ reflected over the x-axis
- $f(-x)$ is $f(x)$ reflected over the y-axis
- $af(x)$ is $f(x)$ with a vertical stretch or shrink

Functions describe the relationship between two variables. A function is continuous if the graph can be drawn without lifting the pencil from the paper. A graph is discontinuous if it has jumps, breaks, or holes in it. Each function, whether continuous or discontinuous, has a distinct domain, range, zero(s), y-intercept, and inverse.

(continued)
Teacher Notes and Elaborations (continued)

The domain is the set of all possible values for the first coordinate of a function. The range of a function is the set of all possible values for the second coordinate of a function.

The x-intercept is the x-coordinate of the point where the graph crosses the x-axis. The y-intercept is the y-coordinate of the point where the graph crosses the y-axis.

The domain of every quadratic function is the set of all real numbers. As a result, the graph of a polynomial function extends infinitely.

What happens to a polynomial function as its domain values get very small and very large is called the end behavior of a polynomial function.

Quadratic equations can be solved by factoring, graphing, using the quadratic formula, and completing the square (only honors level classes learn completing the square). They can have real or complex solutions. Graphing calculators should be used as a primary tool in solving quadratics and aid in visualizing or confirming solutions. Calculators can also be used to determine a quadratic equation that best fits a data set.

The roots or zeros of a function are the x-values of the x-intercepts of the graphs.

To solve a quadratic equation by factoring, completely factor the function, then set each factor equal to zero to solve for x. When you factor a perfect square quadratic \((x^2 + 6x + 9 = (x + 3)^2)\), you get a repeating zero \((x = -3\) occurs twice or has a multiplicity of two). When zeros repeat, they cause the graph to have different shapes at the x-intercept. If a zero occurs once, the graph will simply cut through the axis, but if a zero repeats twice the graph will touch the x-axis, but ‘bounce’ or ‘ricochet’ off the axis without passing through it.

Using the zeros of a quadratic function, a generic equation of the function can be found. All factors of the function are of the form \((x – zero)\). The factors can then be multiplied.

Ex. \(x = 3, -2\)
Factors: \((x – 3), (x + 2)\)
Generic Equation: \((x – 3)(x + 2) = x^2 – x – 6\)

Since the zeros do not reveal any information about reflections of the graph (causing it to open down vs. up) or dilations, the quadratic function produced is the parent graph for all functions that have the given zeros.

The quadratic formula is \(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\).

The discriminant of a quadratic equation, \(b^2 - 4ac\), provides information about the nature and number of roots of the equation. The following table summarizes all possibilities.

<table>
<thead>
<tr>
<th>Value of (b^2 - 4ac)</th>
<th>Discriminant a perfect square?</th>
<th>Nature of Roots</th>
<th>Nature of Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>&gt; 0</td>
<td>Yes</td>
<td>2 real, rational</td>
<td>Intersects x-axis twice</td>
</tr>
<tr>
<td>&gt; 0</td>
<td>No</td>
<td>2 real, irrational</td>
<td>Intersects x-axis twice</td>
</tr>
<tr>
<td>&lt; 0</td>
<td>---</td>
<td>2 imaginary</td>
<td>Doesn’t intersect x-axis</td>
</tr>
<tr>
<td>= 0</td>
<td>---</td>
<td>1 rational</td>
<td>Touches x-axis once</td>
</tr>
</tbody>
</table>

A solution of an equation makes the equation true for a given value or set of values.

The solution of an equation in one variable can be found by graphing each member of the equation separately and noting the x-coordinate of the point of intersection.

(continued)
### Teacher Notes and Elaborations (continued)

Set builder notation is used to represent solutions. For example, if the solution is \( y = 10 \) then in set notation the answer is written \( \{ y \mid y = 10 \} \).

Graphing calculators are powerful tools for solving and confirming algebraic solutions. Practical problems can be interpreted, represented, and solved using equations.

Conjugate is an adjective used to describe two items having features in common but inverses or opposites in some aspect. Number pairs of the form \( a + \sqrt{b} \) and \( a - \sqrt{b} \) are conjugates. Complex solutions occur in pairs (conjugates). This can be explored using the quadratic formula and discussing the meaning/purpose of the \( \pm \) in front of the radical.

### Sample Instructional Strategies and Activities

- Students, working in cooperative groups of four, write a type of equation (absolute value, quadratic, rational, and radical). Collect and redistribute to different groups. Next, students solve, discuss, and graph the solution. Choose spokesperson to present findings to the class.
- As a cooperative learning activity, students will solve the quadratic equation, \( ax^2 + bx + c = 0 \), by using the completion of the square method to derive the quadratic formula. Students, working in cooperative groups, solve \( x^2 - 5x + 6 = 0 \) in three different ways. Compare and contrast the methods and use the graphing calculator to verify the roots of the equation.
- Consider the activity "I have …Who has?" Last card answer will be on the first card. This game can be used for many different types of equations. Each student must work the problems to determine if they have the card with the correct answer.

- The graphing calculator should be integrated throughout the study of polynomials for predicting solutions, determining the reasonableness of solutions, and exploring the behavior of polynomials.
### Systems of Non-Linear Equations

<table>
<thead>
<tr>
<th><strong>Strand:</strong> Equations and Inequalities</th>
<th><strong>Essential Knowledge and Skills</strong></th>
<th><strong>Essential Understandings</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>SOL AII.5</strong></td>
<td>The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to</td>
<td>• Solutions of a nonlinear system of equations are numerical values that satisfy every equation in the system.</td>
</tr>
<tr>
<td></td>
<td>• Predict the number of solutions to a nonlinear system of two equations.</td>
<td>• The coordinates of points of intersection in any system of equations are solutions to the system.</td>
</tr>
<tr>
<td></td>
<td>• Solve a linear-quadratic system of two equations algebraically and graphically.</td>
<td>• Real-world problems can be interpreted, represented, and solved using systems of equations.</td>
</tr>
<tr>
<td></td>
<td>• Solve a quadratic-quadratic system of two equations algebraically and graphically.</td>
<td></td>
</tr>
</tbody>
</table>

**Essential Knowledge and Skills**

- Predict the number of solutions to a nonlinear system of two equations.
- Solve a linear-quadratic system of two equations algebraically and graphically.
- Solve a quadratic-quadratic system of two equations algebraically and graphically.

**Essential Understandings**

- Solutions of a nonlinear system of equations are numerical values that satisfy every equation in the system.
- The coordinates of points of intersection in any system of equations are solutions to the system.
- Real-world problems can be interpreted, represented, and solved using systems of equations.

(continued)
### Systems of Non-Linear Equations (continued)

#### Resources

**Textbook:**
- pg. 577 Solving Quadratic Systems Extension
- 3-1 Graphing Systems of Equations

**HCPS Algebra 2 Online!**:
- Systems of Non-Linear Equations

**DOE ESS Lesson Plans:**
- Nonlinear Systems of Equations (PDF) (Word)

#### Key Vocabulary

- linear-quadratic equations
- quadratic-quadratic equations
- nonlinear systems of equations

#### Essential Questions

- What is a linear function? In what form(s) can it be written?
- What is a quadratic function? In what form(s) can it be written?
- How does a graphing calculator confirm algebraic solutions of quadratic functions?
- What is a real-world example of a non-linear system of equations?
- What are the different ways that the graph of a line and a quadratic can intersect?
- What are the different ways that the graphs of two quadratics can intersect?
- When does a system have no solution?

#### Teacher Notes and Elaborations

An equation in which one or more terms have a variable of degree 2 or higher is called a nonlinear equation. A **nonlinear system of equations** contains at least one nonlinear equation.

A system of equations is a set of two or more equations that use the same variable(s). If the graph of each equation in a system of two variables is a line, then the system is a linear system. Nonlinear systems of equations can be classified as linear-quadratic or quadratic-quadratic. Both systems can be solved algebraically and graphically.

Solutions of linear and nonlinear systems of equations are values that satisfy every equation in the system.

Points of intersection in linear and nonlinear systems are solutions to the system.

Each point of intersection of the graphs of the equations in a system represents a real solution of the system. A **linear-quadratic system** of equations has two solutions, one solution or no solution. A **quadratic-quadratic system** of equations has four solutions, three solutions, two solutions, one solution, or no solution.

Graphing calculators can be used to visualize a nonlinear system of two equations and predict the number of solutions.

Systems can be solved algebraically using substitution or elimination. To solve by substitution it is recommended that students solve each equation for \( y \) (or the non-quadratic variable). Then the two equations can be set equal to each other to solve for \( x \) (which may have multiple values). Once \( x \) is found, plug its value(s) back into one of the original equations to find \( y \).

#### Sample Instructional Strategies and Activities

- Working in cooperative groups, ask students to discuss an appropriate method for solving a given system. Students will...
Sample Instructional Strategies and Activities (continued)

- Verbalize each step needed and discuss the proper steps needed.

- In order to show students the relationship between the different ways in which two second degree equations can intersect and the number of real solutions, have students draw a circle on graph paper and a parabola on patty paper. Tell students such a system can have zero, one, two, three, or four real solutions. Instruct them to hold the circle stationary and move the parabola to illustrate each number of possible solutions.
## Polynomial Functions

<table>
<thead>
<tr>
<th>Strand: Functions</th>
</tr>
</thead>
</table>

### SOL AII.6

The student will recognize the general shape of function (square root, cube root, and polynomial) families and will convert between graphic and symbolic forms of functions. A transformational approach to graphing will be employed.

### SOL AII.7

The student will investigate and analyze functions algebraically and graphically. Key concepts include:
- Domain and range, including limited and discontinuous domains and ranges;
- Zeros;
- \( x \)- and \( y \)-intercepts;
- Intervals in which a function is increasing or decreasing;
- End behavior;

Graphing calculators will be used as a tool to investigate the shapes and behaviors of these functions.

### Essential Knowledge and Skills

The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to:

- Recognize graphs of parent functions.
- Given a transformation of a parent function, identify the graph of the transformed function.
- Given the equation and using a transformational approach, graph a function.
- Given the graph of a function, identify the parent function.
- Given the graph of a function, identify the transformations that map the pre-image to the image in order to determine the equation of the image.
- Using a transformational approach, write the equation of a function given its graph.
- Identify the domain, range, zeros, and intercepts of a function presented algebraically or graphically.
- Describe restricted/discontinuous domains and ranges.
- Given the graph of a function, identify intervals on which the function is increasing and decreasing.
- Describe the end behavior of a function.

### Essential Understandings

- The graphs/equations for a family of functions can be determined using a transformational approach.
- Transformations of graphs include translations, reflections, and dilations.
- A parent graph is an anchor graph from which other graphs are derived with transformations.
- Functions may be used to model real-world situations.
- The domain and range of a function may be restricted algebraically or by the real-world situation modeled by the function.
- A function can be described on an interval as increasing, decreasing, or constant.
- End behavior describes a function as \( x \) approaches positive and negative infinity.
- A zero of a function is a value of \( x \) that makes \( f(x) \) equal zero.
## Resources

**Textbook:**
- 6-1 Polynomial Functions
- 2-6 Vertical and Horizontal Translations
- pg. Geometric Transformation Extension
- pg. 306 End Behavior Extension

**HCPS Algebra 2 Online!**:
- Function Families
- Characteristics of Functions

**DOE ESS Lesson Plans:**
- Transformational Graphing (PDF) (Word)
- Functions: Domain, Range, End Behavior, Increasing, Decreasing (PDF) (Word)
- Composition of Functions (PDF) (Word)
- Inverse Functions (PDF) (Word)

## Key Vocabulary

- continuous
- cube root function
- cubic function
- decreasing
- dilation
- discontinuous
- domain
- end behavior
- family of functions
- increasing
- intercepts (x and y)
- multiplicities
- parent function
- polynomial function
- quartic function
- range
- reflection
- repeating zeros
- square root function
- transformations of graphs
- translation (vertical/horizontal)

## Essential Questions

- What is the transformational approach to graphing?
- What is the connection between the algebraic and graphical representation of a transformation?
- What is a function?
- What is the relationship between domain and range?
- What is the relationship between a function and its inverse?
- What operations can be performed on functions?
- What is the relationship between the degree of a function, the graph of a function, and the number of zeros of a function?
- How can the calculator be used to investigate the shape and behavior of polynomial functions?
- How are the x- and y-intercepts determined?
- What is meant by the end behavior of a function?
- Describe the characteristics of the graphs of odd-degree and even-degree polynomial functions whose leading coefficients are positive. How does negative leading coefficients change each of these?
- What is meant by the turning points of a function and how are they found?

## Teacher Notes and Elaborations

A function is a correspondence in which values of one variable determine the values of another. It is a rule of correspondence between two sets such that there is a unique element in one set assigned to each element in the other.

A *polynomial function* is a function of one variable whose exponents are natural numbers. The degree of a polynomial function determines
Polynomial and Radical Functions (continued)

**Teacher Notes and Elaborations (continued)**

its graphing behavior. A polynomial function is linear, quadratic, cubic, quartic, etc., according to its degree, 1, 2, 3, 4, …, respectively. The degree of the polynomial will help determine the graph of the polynomial function.

The graphs and/or equations for a family of functions can be determined using a transformational approach. A family of functions is a group of functions with common characteristics. A parent function is the simplest function with these characteristics. A parent function and one or more transformations make up a family of functions. Shapes and behavior of graphs of polynomials can be determined by analyzing transformations of parent functions.

A quadratic function is of degree two, a cubic function is of degree three and a quartic function is of degree four.

<table>
<thead>
<tr>
<th>Quadratic</th>
<th>Cubic</th>
<th>Quartic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = 2x^2 - 3 )</td>
<td>( f(x) = 2x^3 - 3x - 1 )</td>
<td>( f(x) = x^4 - 2x^3 + 2x + 1 )</td>
</tr>
</tbody>
</table>

The points on the graph of a polynomial function that correspond to local maxima and local minima are called turning points. A turning point in a graph occurs when the function values change from increasing to decreasing or vice versa and indicates either a local maximum or a local minimum. The number of turning points in a graph is no greater than 1 less than the degree of the function.

The domain of every polynomial function is the set of all real numbers. As a result, the graph of a polynomial function extends infinitely. What happens to a polynomial function as its domain values get very small and very large is called the end behavior of a polynomial function.

If a polynomial function is written in standard form, \( f(x) = a_nx^n + a_{n-1}x^{n-1} + … + a_1x + a_0 \), the leading coefficient is \( a_n \). The leading coefficient is the coefficient of the term of greatest degree in the polynomial. \( a_n \) and \( n \) determine the end behavior of the graph of any polynomial function.

- When the degree of the function is odd and the leading coefficient is positive, the graph falls on the left and rises on the right.
- When the degree of the function is odd and the leading coefficient is negative, the graph rises on the left and falls on the right.
- When the degree of the function is even and the leading coefficient is positive, the graph rises on the left and on the right.
- When the degree of the function is even and the leading coefficient is negative, the graph falls on the left and the right.

(continued)
A transformation of a function is an alteration of the function rule that results in an alteration of its graph.

If \( y = f(x) \), then \( y = f(x) + k \) gives a **vertical translation** of the graph of \( f \). The translation is \( k \) units up for \( k > 0 \) and \( |k| \) units down for \( k < 0 \). If \( y = f(x) \), then \( y = f(x - h) \) gives a **horizontal translation** of the graph of \( f \). The translation is \( h \) units to the right for \( h > 0 \) and \( |h| \) units to the left for \( h < 0 \). Note: the standard form for a horizontal translation contains a subtraction sign. Thus when determining the direction of the shift, students need to separate the value of \( h \) from the operation seen in front of it (it is the ‘opposite’ of the obvious value).

If \( y = f(x) \), then \( y = -f(x) \) gives a **reflection** of the graph of \( f \) across the \( x \)-axis. If \( y = f(x) \), then \( y = f(-x) \) gives a reflection of the graph of \( f \) across the \( y \)-axis.

A curve’s shape is determined by the rule or relation which defines it. Very often multiplication or division is involved. A line’s slope and a parabola’s stretch or compression is attributed to **dilation** – expansion, growing or shrinking, multiplication. If \( y = f(x) \) , then \( y = af(x) \) gives a vertical stretch or vertical compression of the graph of \( f \). If \( a > 1 \), the graph is stretched vertically by a factor of \( a \). If \( 0 < a < 1 \), the graph is compressed vertically by a factor of \( a \).

Rules recap:
- \( f(x) + a \) is \( f(x) \) shifted upward \( a \) units
- \( f(x) - a \) is \( f(x) \) shifted downward \( a \) units
- \( f(x + a) \) is \( f(x) \) shifted left \( a \) units
- \( f(x - a) \) is \( f(x) \) shifted right \( a \) units
- \( -f(x) \) is \( f(x) \) reflected over the \( x \)-axis
- \( f(-x) \) is \( f(x) \) reflected over the \( y \)-axis
- \( af(x) \) is \( f(x) \) with a vertical stretch or shrink

Functions describe the relationship between two variables. A function is **continuous** if the graph can be drawn without lifting the pencil from the paper. A graph is **discontinuous** if it has jumps, breaks, or holes in it. Each function, whether continuous or discontinuous, has a distinct domain, range, zero(s), \( y \)-intercept, and inverse.

The **domain** is the set of all possible values for the first coordinate of a function. The **range** of a function is the set of all possible values for the second coordinate of a function.

The **\( x \)-intercept** is the \( x \)-coordinate of the point where the graph crosses the \( x \)-axis and has the ordered pair \((x, 0)\). The **\( y \)-intercept** is the \( y \)-coordinate of the point where the graph crosses the \( y \)-axis and has the ordered pair \((0, y)\).

A function is **increasing** on an interval if its graph always rises as it
### Teacher Notes and Elaborations (continued)

Polynomial Functions (continued)

A function is constant on an interval if its graph is horizontal over the interval. For any \( x_1 \) and \( x_2 \) in the interval, where \( x_1 < x_2 \), then \( f(x_1) = f(x_2) \)

Example:

![Graph](image)

This function is constant on the interval \([-5, 2]\), decreasing on the interval \((2, 3)\), and increasing on the interval \([3, \infty)\).

**Honors/Pre-AP Extension**

If \( y = f(x) \), then \( y = f(bx) \) gives a horizontal stretch or horizontal compression of the graph of \( f \). If \( b > 1 \), the graph is compressed horizontally by a factor of \( \frac{1}{b} \). If \( 0 < b < 1 \), the graph is stretched horizontally by a factor of \( \frac{1}{b} \).

### Sample Instructional Strategies and Activities

- The students are divided into small groups and given graph representations of the following functions: absolute value, square root, cube root, rational, polynomial, exponential, and logarithmic. They must identify the graph representations and explain the transformations from the basic graph of each function. Each group must verify their answers on the graphing calculator.

- Students, working in small groups, will be given several selective polynomial functions. They will use the graphing calculator to categorize these functions. Next, each group will draw conclusions about the general shape, end behavior, zeros, and \( y \)-intercept of the functions.

- On cards write 20 functions beginning with \( a(x) \), \( b(x) \), \( c(x) \) .... Mix the cards and select two at random. Have students write the composition of the two functions [e.g., \( a(d(x)) \)]. Using the same functions have students reverse the order [e.g., \( (d(a(x))) \)]. Is composition of functions commutative? Use the same cards and have students perform the four operations.
### Rational Expression and Equations

**Strand:** Expressions and Operations; Equations and Inequalities

#### SOL AII.1a
The student, given rational expressions, will add, subtract, multiply, divide, and simplify rational algebraic expressions.

#### SOL AII.4c
The student will solve, algebraically and graphically, equations containing rational algebraic expressions.

**Essential Knowledge and Skills**
The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to

- Add, subtract, multiply, and divide rational algebraic expressions.
- Simplify a rational algebraic expression with common monomial or binomial factors.
- Recognize a complex algebraic fraction, and simplify it as a quotient or product of simple algebraic fractions.
- Solve equations containing rational algebraic expressions with monomial or binomial denominators algebraically and graphically.
- Verify possible solutions to an equation containing rational expressions.
- Apply an appropriate equation to solve a real-world problem.

**Essential Understandings**
- Computational skills applicable to numerical fractions also apply to rational expressions involving variables.
- A relationship exists among arithmetic complex fractions, algebraic complex fractions, and rational numbers.
- The process of solving rational equations can lead to extraneous solutions.
- Equations can be solved in a variety of ways.
- Set builder notation may be used to represent solution sets of equations and inequalities.
- Real-world problems can be interpreted, represented, and solved using equations and inequalities.

(continued)
### Resources

**Textbook:**
- 9-4 Rational Expressions
- 9-5 Adding and Subtracting Rational Expressions
- 9-6 Solving Rational Equations

**HCPS Algebra 2 Online!**
- Rational Expressions (PDF) (Word)
- Rational Equations (PDF) (Word)

### Key Vocabulary
- complex fraction
- rational expression
- restricted variable

### Essential Questions
- What is a rational expression?
- How is a rational expression simplified?
- What is a real-world example for a quadratic, absolute value, radical, or rational equation?
- When working with a real-world problem, how are solution(s) verified?
- How is an equation containing rational expressions solved?
- What values must be eliminated from the solution set of a rational equation?

### Teacher Notes and Elaborations

A *rational expression* is written as a quotient of polynomials in simplest form; the divisor is never zero. Equations containing rational expressions can be solved by finding the least common denominator.

Rational expressions must be simplified. A simplified expression meets the following conditions:
1. It has no negative exponents.
2. It has no fractional exponents in the denominator.
3. It is not a complex fraction.

Rational expressions can be added, subtracted, multiplied, and divided.

To divide a polynomial by a binomial either factor or use long division. Factoring and simplifying is the preferred method (and creates a rational expression) but does not always work. In these cases, long division is used.

#### Factoring Example:

\[
\frac{x^2 + 8x + 15}{x + 3} = \frac{(x + 5)(x + 3)}{(x + 3)} = \frac{(x + 5)(x + 3)}{(x + 3)} = (x + 5)
\]

#### Long Division Example:

\[
\frac{a^2 - 4a - 6}{a + 2} \quad \rightarrow \quad a + 2 \left[ a^2 - 4a - 6 \right] \quad \rightarrow \quad a - 6 + \frac{6}{a + 2}
\]

\[
-(a^2 + 2a) \quad \rightarrow \quad \frac{-(a^2 - 2a) - 6}{a + 2} \quad \rightarrow \quad \frac{-(a^2 - 2a) - 6}{a + 2} + 6
\]
Teacher Notes and Elaborations (continued)

A complex fraction is a fraction that has a fraction in its numerator and/or denominator. A rational expression is a polynomial or the quotient of two polynomials. The denominator cannot be 0 (this is called a restricted variable). Rational expressions written as complex fractions can be written as a quotient or product of simple fractions.

When solving equations it is important to check possible solutions in the original equation as one or more may be an extraneous solution. An extraneous solution is a solution of an equation derived from an original equation that is not a solution of the original equation. Absolute value, radical, and rational equations may have extraneous solutions.

Rational equations can be solved by either canceling out each denominator by multiplying each side of the equation by the least common denominator or by building up the denominators to create a common denominators. Once each term has the common denominator, you can create a polynomial equation from the numerators.

A solution of an equation makes the equation true for a given value or set of values. Equations containing rational expressions can be solved in a variety of ways.

The solution of an equation in one variable can be found by graphing each member of the equation separately and noting the x-coordinate of the point of intersection.

Set builder notation is used to represent solutions. For example, if the solution is \( y = 10 \) then in set notation the answer is written \( \{ y : y = 10 \} \).

Graphing calculators are powerful tools for solving and confirming algebraic solutions. Practical problems can be interpreted, represented, and solved using equations.

Sample Instructional Strategies and Activities

- Consider the activity "I have …Who has?" Last card answer will be on the first card. This game can be used for many different types of equations. Each student must work the problems to determine if they have the card with the correct answer.
- Students will be divided into cooperative groups of four and be given a pair of rational expressions to be added, to be subtracted, to be multiplied, and to be divided. They will, in turn, choose an operation, perform it and then explain it to the other group members. As a part of this process, the group members will correct each other's mistakes.
- Students, working in cooperative groups of four, write a type of equation (absolute value, quadratic, rational, and radical). Collect and redistribute to different groups. Next, students solve, discuss, and graph the solution. Choose spokesperson to present findings to the class.
### Exponential and Logarithmic Functions

**Strand:** Functions

**SOL AII.6** The student will recognize the general shape of function (rational, exponential, and logarithmic) families and will convert between graphic and symbolic forms of functions. A transformational approach to graphing will be employed.

**SOL AII.7** The student will investigate and analyze functions algebraically and graphically. Key concepts include
- a) domain and range, including limited and discontinuous domains and ranges;
- b) zeros;
- c) $x$- and $y$-intercepts;
- d) intervals in which a function is increasing or decreasing;
- e) asymptotes;
- f) end behavior;
- g) inverse of a function
- h) composition of multiple functions.

Graphing calculators will be used as a tool to investigate the shapes and behaviors of these functions.

#### Essential Knowledge and Skills
- The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to
  - Recognize graphs of parent functions.
  - Given a transformation of a parent function, identify the graph of the transformed function.
  - Given the equation and using a transformational approach, graph a function.
  - Given the graph of a function, identify the parent function.
  - Given the graph of a function, identify the transformations that map the preimage to the image in order to determine the equation of the image.
  - Using a transformational approach, write the equation of a function given its graph.
  - Identify the domain, range, zeros, and intercepts of a function presented algebraically or graphically.
  - Describe restricted/discontinuous domains and ranges.
  - Given the graph of a function, identify intervals on which the function is increasing and decreasing.
  - Find the equations of vertical and horizontal asymptotes of functions.
  - Describe the end behavior of a function.
  - Investigate exponential and logarithmic functions, using the graphing calculator.
  - Convert between logarithmic and exponential forms of an equation with bases consisting of natural numbers.
  - Find the inverse of a function.
  - Graph the inverse of a function as a reflection across the line $y = x$.
  - Find the composition of two functions.
  - Use composition of functions to verify two functions are inverses.

#### Essential Understandings
- The graphs/equations for a family of functions can be determined using a transformational approach.
- Transformations of graphs include translations, reflections, and dilations.
- A parent graph is an anchor graph from which other graphs are derived with transformations.
- Functions may be used to model real-world situations.
- The domain and range of a function may be restricted algebraically or by the real-world situation modeled by the function.
- A function can be described on an interval as increasing, decreasing, or constant.
- Asymptotes may describe both local and global behavior of functions.
- End behavior describes a function as $x$ approaches positive and negative infinity.
- A zero of a function is a value of $x$ that makes $f(x)$ equal zero.
- Exponential ($y = a^x$) and logarithmic ($y = \log_a x$) functions are inverses of each other.
- If $(a, b)$ is an element of a function, then $(b, a)$ is an element of the inverse of the function.
- Functions can be combined using composition of functions.

*Return to Course Outline.*
Exponential and Logarithmic Functions (continued)

Resources
Textbook:
- 8-3 Logarithmic Functions as Inverses
- 8-1 Exploring Exponential Models
- 8-2 Properties of Exponential Functions
- 9-3 Rational Functions and their Graphs
- 7-6 Function Operations
- 7-7 Inverse Relations and Functions

HCPS Algebra 2 Online!:
- Function Families
- Characteristics of Functions

DOE ESS Lesson Plans:
- Rational Functions: Intercepts, Asymptotes, and Discontinuity (PDF) (Word)
- Transformational Graphing – Functions (PDF) (Word)
- Functions: Domain, Range, End Behavior, Increasing, Decreasing
  - (PDF) (Word)

Key Vocabulary
- asymptotes
- composition of functions
- continuous
- decreasing
dilation
discontinuous
domain
end behavior
exponential function
family of functions
increasing
intercepts (x and y)
inverse of a function
logarithmic function
parent function
rational function
range
reflection
transformations of graphs
translation (vertical/horizontal)
zero(s)

Essential Questions
- What are different representations of functions?
- What is the transformational approach to graphing?
- What is the connection between the algebraic and graphical representation of a transformation?
- What is the relationship between exponential and logarithmic functions?
- How can the calculator be used to investigate these functions (rational, exponential, and logarithmic)?
- What is a function?
- What is the relationship between domain and range?
- What is the relationship between a function and its inverse?
- What is a zero of a function?
- What operations can be performed on functions?
- What is the relationship between exponential and logarithmic functions?
- How can the calculator be used to investigate the shape and behavior of exponential, and logarithmic functions?
- How are the x- and y-intercepts determined?
- What is an asymptote?
- What role do asymptotes have in graphing functions?
- What is meant by the end behavior of a function?
- How can a hole in the graph of a function be determined?
- What is meant by the turning points of a function and how are they found?
- How do asymptotes affect the domain or range of a function?
- How do you find the inverse of a function?
- What is meant by composition of functions?
**Teacher Notes and Elaborations**

An exponential function is a function of the form \( y = a^x \), where \( a \) is a positive constant not equal to one. Population growth and viral growth are among examples of exponential functions. Logarithmic functions are inverses of exponential functions and have the form \( y = \log_a x \).

Given the exponential equation \( b^x = y \), \( b \) is the base, \( x \) is the exponent and \( y \) is the value. To solve the equation about for \( x \), we can use logs.

\[
\begin{align*}
b^x = y & \iff x = \log_b y \\
(\text{read } x \text{ equals log base } b \text{ of } y)
\end{align*}
\]

A function is a correspondence in which values of one variable determine the values of another. It is a rule of correspondence between two sets such that there is a unique element in one set assigned to each element in the other.

The graphs and/or equations for a family of functions can be determined using a transformational approach. A family of functions is a group of functions with common characteristics. A parent function is the simplest function with these characteristics. A parent function and one or more transformations make up a family of functions. Shapes and behavior of graphs of polynomials can be determined by analyzing transformations of parent functions.

The following is **one** example of a parent function and family of functions.

**Parent Function**

\[ f(x) = 2^x \]

**Family of Functions**

\[
\begin{align*}
f(x) &= 2^{x-4} + 1 \\
f(x) &= -2^{x+1} - 3
\end{align*}
\]

A rational function \( f(x) \) can be written as \( f(x) = \frac{P(x)}{Q(x)} \), where \( P(x) \) and \( Q(x) \) are polynomial functions and \( Q(x) \neq 0 \).

---

**Rational**  \( f(x) = \frac{4}{x-3} - 1.5 \)

**Exponential**  \( f(x) = 2^x \)

**Logarithmic**  \( f(x) = \log_2 x \)

A function is continuous if the graph can be drawn without lifting the pencil from the paper. A graph is discontinuous if it has jumps, breaks, or holes in it. Rational functions are typically discontinuous. Rational, exponential, and logarithmic functions are asymptotic. An asymptote is a line that a graph approaches as \( x \) or \( y \) increases in absolute value. An asymptote of a curve is a line such that the distance between the curve and the line approaches zero as it tends to infinity. Rational, exponential, and logarithmic functions all have asymptotes.

A transformation of a function is an alteration of the function rule that results in an alteration of its graph.

(continued)
Exponential and Logarithmic Functions (continued)

If \( y = f(x) \), then \( y = f(x) + k \) gives a vertical translation of the graph of \( f \). The translation is \( k \) units up for \( k > 0 \) and \( |k| \) units down for \( k < 0 \). If \( y = f(x) \), then \( y = f(x - h) \) gives a horizontal translation of the graph of \( f \). The translation is \( h \) units to the right for \( h > 0 \) and \( |h| \) units to the left for \( h < 0 \). Note: the standard form for a horizontal translation contains a subtraction sign. Thus when determining the direction of the shift, students need to separate the value of \( h \) from the operation seen in front of it (it is the ‘opposite’ of the obvious value). If the function is given as \( y = f(x + h) \), then the value of \( h \) is actually negative \( y = f(x - (-h)) \).

If \( y = f(x) \), then \( y = -f(x) \) gives a reflection of the graph of \( f \) across the \( x \)-axis. If \( y = f(x) \), then \( y = f(-x) \) gives a reflection of the graph of \( f \) across the \( y \)-axis.

A graph’s shape is determined by the rule or relation which defines it. Very often multiplication or division is involved. A line’s slope and a parabola’s stretch or compression is attributed to dilation – expansion, growing or shrinking, multiplication. If \( y = f(x) \), then \( y = af(x) \) gives a vertical stretch or vertical shrink of the graph of \( f \). If \( a > 1 \), the graph is stretched vertically by a factor of \( a \). If \( 0 < a < 1 \), the graph is compressed vertically by a factor of \( a \).

Rules recap:

- \( f(x) + a \) is \( f(x) \) shifted upward \( a \) units
- \( f(x) - a \) is \( f(x) \) shifted downward \( a \) units
- \( f(x + a) \) is \( f(x) \) shifted left \( a \) units
- \( f(x - a) \) is \( f(x) \) shifted right \( a \) units
- \( -f(x) \) is \( f(x) \) reflected over the \( x \)-axis
- \( f(-x) \) is \( f(x) \) reflected over the \( y \)-axis
- \( af(x) \) is \( f(x) \) with a vertical stretch or shrink

The domain is the set of all possible values for the first coordinate of a function. The range of a function is the set of all possible values for the second coordinate of a function. Asymptotic functions typically have limited domains and ranges.

For a rational function, if \( x - b \) is a factor of the numerator and the denominator of a rational function, then there is a hole in the graph of the function when \( x = b \) unless \( x = b \) is a vertical asymptote. In either case, you must exclude \( b \) from the domain of the function.

The \( x \)-intercept is the \( x \)-coordinate of the point where the graph crosses the \( x \)-axis. The \( y \)-intercept is the \( y \)-coordinate of the point where the graph crosses the \( y \)-axis.

What happens to a function as its domain values get very small and very large is called the end behavior of a polynomial function. Due to the asymptotic nature of rational, exponential, and logarithmic functions, the end behavior of these functions, unlike that of polynomial functions, may approach a value other than \( \pm \) infinity.

The inverse of a function consisting of the ordered pairs \((x, y)\) is the set of all ordered pairs \((y, x)\). The domain of the inverse is the range of the original relation. The range of the inverse is the domain of the original relation.

Graphs of functions that are inverses of each other are reflections across the line \( y = x \). Exponential and logarithmic functions, inverses of each other, are either strictly increasing or strictly decreasing. Exponential and logarithmic functions have asymptotes.

A function is increasing on an interval if its graph always rises as it moves from left to right over the interval. It is decreasing on an interval if its graph always falls as it moves from left to right over the
## Exponential and Logarithmic Functions (continued)

**Teacher Notes and Elaborations**

Exponential and logarithmic functions do not have turning points and are always either increasing or decreasing.

For a function \( y = f(x) \):
- When \( x_1 < x_2 \), then \( f(x_1) \leq f(x_2) \) the function is increasing.
- When \( x_1 > x_2 \), then \( f(x_1) \geq f(x_2) \) the function is decreasing.

Functions can be combined through addition, subtraction, multiplication, division, and composition.

**Composition of functions** refers to the forming of a new function \( h \) (the composite function) from given functions \( g \) and \( f \) by the rule:
\[
 h(x) = g(f(x)) \quad \text{or} \quad h(x) = (g \circ f)(x)
\]
for all \( x \) in the domain of \( f \) for which \( f(x) \) is in the domain of \( g \). This function is read as “\( g \) of \( f \)”. The order in which functions are combined is important.

The composition of a function and its inverse is the identity function.

The **inverse of a function** consisting of the ordered pairs \((x, y)\) is the set of all ordered pairs \((y, x)\). The domain of the inverse is the range of the original relation. The range of the inverse is the domain of the original relation. Graphs of functions that are inverses of each other are reflections across the line \( y = x \).

### Sample Instructional Strategies and Activities

- The students are divided into small groups and given graph representations of the following functions: absolute value, square root, cube root, rational, polynomial, exponential, and logarithmic. They must identify the graph representations and explain the transformations from the basic graph of each function. Each group must verify their answers on the graphing calculator.
- Use compound interest, population growth, and/or rate of decay as examples of exponential functions.

Students, working in small groups, will be given several selective logarithmic and exponential functions. They will use the graphing calculator to categorize these functions. Next, each group will draw conclusions about the general shape and end behavior of the functions.
### Sequences and Series

**Strand**: Functions

<table>
<thead>
<tr>
<th>Essential Knowledge and Skills</th>
<th>Essential Understandings</th>
</tr>
</thead>
</table>
| SOL AII.2 The student will investigate and apply the properties of arithmetic and geometric sequences and series to solve real-world problems, including writing the first \( n \) terms, finding the \( n^{th} \) term, and evaluating summation formulas. Notation will include \( \Sigma \) and \( a_n \). | - Sequences and series arise from real-world situations.  
- The study of sequences and series is an application of the investigation of patterns.  
- A sequence is a function whose domain is the set of natural numbers.  
- Sequences can be defined explicitly and recursively. |

**Essential Knowledge and Skills**

The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to

- Distinguish between a sequence and a series.  
- Generalize patterns in a sequence using explicit and recursive formulas.  
- Use and interpret the notations \( \Sigma \), \( n \), \( n^{th} \) term, and \( a_n \).  
- Given the formula, find \( a_n \) (the \( n^{th} \) term) for an arithmetic or a geometric sequence.  
- Given formulas, write the first \( n \) terms and find the sum, \( S_n \), of the first \( n \) terms of an arithmetic or geometric series.  
- Given the formula, find the sum of a convergent infinite series.  
- Model real-world situations using sequences and series.

(continued)
### Sequences and Series (continued)

#### Resources

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<td>11-2 Arithmetic Sequences</td>
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<td>11-3 Geometric Sequences</td>
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<td>11-4 Arithmetic Series</td>
</tr>
<tr>
<td>pg. 613 Geometry and Infinite Series Investigation</td>
</tr>
<tr>
<td>11-5 Geometric Series</td>
</tr>
</tbody>
</table>

**HCPS Algebra 2 Online!**:

- Sequences and Series

**DOE ESS Lesson Plans**:

- Arithmetic and Geometric Sequences and Series (PDF) (Word)

#### Key Vocabulary

<table>
<thead>
<tr>
<th><strong>arithmetic sequence</strong></th>
<th><strong>geometric sequence</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>common difference</strong></td>
<td><strong>recursive formula</strong></td>
</tr>
<tr>
<td><strong>common ratio</strong></td>
<td><strong>sequence</strong></td>
</tr>
<tr>
<td><strong>convergent</strong></td>
<td><strong>series</strong></td>
</tr>
<tr>
<td><strong>divergent</strong></td>
<td><strong>sigma notation (summation notation)</strong></td>
</tr>
<tr>
<td><strong>explicit formula</strong></td>
<td>---</td>
</tr>
</tbody>
</table>

#### Essential Questions

- What is the difference between a series and a sequence?
- What is the difference between arithmetic and geometric sequences and series?
- What is Sigma notation (\(\Sigma\))?
- What real-world situations use sequences and series?

#### Teacher Notes and Elaborations

A *sequence* is a mathematical pattern of numbers. A *series* is an indicated sum of a sequence of terms. Both sequences and series may be finite or infinite.

Two types of sequences are arithmetic and geometric. An *arithmetic sequence* is a sequence in which each term, after the first term, is found by repeated addition of a constant, called the *common difference* to the previous term. A *geometric sequence* is a sequence in which each term, after the first term, can be found by multiplying the preceding term by a nonzero constant, called the *common ratio*. The terms between any two nonconsecutive terms of a geometric/arithmetic sequence are called the geometric/arithmetic means.

A series is the expression for the sum of the terms of a sequence. Finite sequences and series have terms that can be counted individually from 1 to a final whole number \(n\). Infinite sequences and series continue without end. Infinite sequences or series are indicated with ellipsis points (3 dots indicating the missing part of a statement).

**Example**:  
Finite sequence \(4, 8, 12, 16, 20\)  
Finite series \(4 + 8 + 12 + 16 + 20\)  
Infinite sequence \(2, 7, 12, 17, \ldots\)  
Infinite series \(2 + 7 + 12 + 17 + \ldots\)

There are formulas that assist in determining elements of the sequence \(a_n\) or sum of a series. The *sigma notation* or *summation notation* (\(\Sigma\)) is one way of writing a series. \(\Sigma\) is the Greek letter sigma, the equivalent of the English letter \(S\) (for summation). Use limits to indicate how many terms are added. Limits are the least and greatest integral values of \(n\).

**Example**:  
Use summation notation to write the series \(4 + 8 + 12 + \ldots\) for 23 terms.

(continued)
Teacher Notes and Elaborations (continued)

\[ \sum_{n=1}^{23} (4n) \] where 23 is the upper limit (greatest value of \( n \)), 1 is the lower limit (least value of \( n \)) and the explicit formula for the series is \( 4n \).

A recursive formula for a sequence describes how to find the \( n^{th} \) term from the terms before it. A recursive formula defines the terms in a sequence by relating each term to the ones before it. An explicit formula expresses the \( n^{th} \) term in terms of \( n \).

### Arithmetic Sequence Formulas

<table>
<thead>
<tr>
<th>Recursive Formula</th>
<th>Explicit Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 ) is a given value, ( a_n = a_{n-1} + d )</td>
<td>( a_n = a_1 + (n-1)d )</td>
</tr>
</tbody>
</table>

In these formulas, \( a_n \) is the \( n^{th} \) term, \( a_1 \) is the first term, \( n \) is the number of the term, and \( d \) is the common difference.

### Geometric Sequence Formulas

<table>
<thead>
<tr>
<th>Recursive Formula</th>
<th>Explicit Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 ) is a given value, ( a_n = a_{n-1} \cdot r )</td>
<td>( a_n = a_1 \cdot r^{n-1} )</td>
</tr>
</tbody>
</table>

In these formulas, \( a_n \) is the \( n^{th} \) term, \( a_1 \) is the first term, \( n \) is the number of the term, and \( r \) is the common ratio.

In some cases, an infinite geometric series can be evaluated. When \( |r| < 1 \), the series is convergent, or gets closer and closer to the sum \( S \). When \( |r| \geq 1 \), the series is divergent, or approaches no limit.

Real-world applications such as fractals, growth, tax, and interest are solved using sequences and series.

### Sample Instructional Strategies and Activities

- Model real-life situations with arithmetic sequences, such as increasing lengths of rows in some concert halls.
- Model real-life situations with geometric sequences, such as revenues that increase at a constant percent.
- Model a real-life situation with geometric series, such as total distance traveled by a bouncing ball.
- Assign groups of students to make up a geometric series and write it in sigma notation. Groups exchange problems and find the sum.

### Sum of a Finite Arithmetic Series

The \( S_n \) of a finite arithmetic series \( a_1 + a_2 + a_3 + \ldots + a_n \) is

\[ S_n = \frac{n}{2} (a_1 + a_n) \]

where \( a_1 \) is the first term, \( a_n \) is the \( n^{th} \) term, and \( n \) is the number of terms. Another formula for the sum of a finite arithmetic series is

\[ S_n = \frac{n}{2} [2a_1 + (n-1)d] \]

where \( a_1 \) is the first term, \( a_n \) is the \( n^{th} \) term, and \( n \) is the number of terms.

### Sum of a Finite Geometric Series

The \( S_n \) of a finite geometric series \( a_1 + a_2 + a_3 + \ldots + a_n \), \( r \neq 1 \), is

\[ S_n = \frac{a_1(1-r^n)}{1-r} \]

where \( a_1 \) is the first term, \( r \) is the common ratio, and \( n \) is the number of terms.

### Sum of an Infinite Geometric Series

An infinite geometric series with \( |r| < 1 \) converges to the sum

\[ S_\infty = \frac{a_1}{1-r} \]

where \( a_1 \) is the first term and \( r \) is the common ratio.
### Sequences and Series (continued)

**Sample Instructional Strategies and Activities (continued)**

- Return the results to the original group to check accuracy.
- For a geometric series, students are offered a penny for the first day of the month, to be doubled every day thereafter for the month. Have students determine the amount on the 30\textsuperscript{th} day and the total for the thirty days.
- Compare and contrast between arithmetic and geometric series: Students are offered two jobs one with a constant annual raise versus a percentage raise. Which job will have the highest salary after twenty years?

*Return to Course Outline*
## Normal Distribution and Z-scores

**Strand:** Statistics

**Essential Knowledge and Skills**
The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to:
- Identify the properties of a normal probability distribution.
- Describe how the standard deviation and the mean affect the graph of the normal distribution.
- Compare two sets of normally distributed data using a standard normal distribution and z-scores.
- Represent probability as area under the curve of a standard normal probability distribution.
- Use the graphing calculator or a standard normal probability table to determine probabilities or percentiles based on z-scores.

**Essential Understandings**
- A normal distribution curve is a symmetrical, bell-shaped curve defined by the mean and the standard deviation of a data set. The mean is located on the line of symmetry of the curve.
- Areas under the curve represent probabilities associated with continuous distributions.
- The normal curve is a probability distribution and the total area under the curve is 1.
- For a normal distribution, approximately 68 percent of the data fall within one standard deviation of the mean, approximately 95 percent of the data fall within two standard deviations of the mean, and approximately 99.7 percent of the data fall within three standard deviations of the mean.
- The mean of the data in a standard normal distribution is 0 and the standard deviation is 1.
- The standard normal curve allows for the comparison of data from different normal distributions.
- A z-score is a measure of position derived from the mean and standard deviation of data.
- A z-score expresses, in standard deviation units, how far an element falls from the mean of the data set.
- A z-score is a derived score from a given normal distribution.
- A standard normal distribution is the set of all z-scores

(continued)
Normal Distribution and Z-scores (continued)

<table>
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<th>Resources</th>
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<tbody>
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<tr>
<td>• Normal Distributions (PDF) (Word)</td>
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</table>

<table>
<thead>
<tr>
<th>Key Vocabulary</th>
</tr>
</thead>
<tbody>
<tr>
<td>area under a curve</td>
</tr>
<tr>
<td>mean</td>
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<tr>
<td>normal distribution curve</td>
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<tr>
<td>normal probability distribution</td>
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<tr>
<td>population</td>
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<tr>
<td>standard deviation</td>
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<tr>
<td>standard normal curve</td>
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<td>percentile</td>
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<td>z-score</td>
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<table>
<thead>
<tr>
<th>Essential Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>• What is a normal distribution curve and how is the graph constructed?</td>
</tr>
<tr>
<td>• How can the amount of data that lies within 1, 2, 3, or k standard deviations of the mean be found?</td>
</tr>
<tr>
<td>• How does the standard normal distribution curve correspond to probability?</td>
</tr>
<tr>
<td>• How can the area under the standard normal curve be found?</td>
</tr>
<tr>
<td>• How is a standard normal probability table used and applied in problem solving?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Teacher Notes and Elaborations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Statistics is the science of collecting, analyzing, and drawing conclusions from data. Methods for organizing and summarizing data make up the branch of statistics called descriptive statistics.</td>
</tr>
<tr>
<td>Measures of dispersion indicate the extent to which values are spread around a central value such as the mean when doing standard deviation or the median when doing box-and-whisker plots.</td>
</tr>
<tr>
<td>Differences from the mean, ( x - \mu ), are called deviations. The deviation of an entry ( x ) in a population data set is the difference between the entry and the mean ( \mu ) of the data set. The mean is the balance point of the distribution, so the set of deviations from the mean will always add to zero.</td>
</tr>
</tbody>
</table>

The following information is taken from the VDOE Technical Assistance Document – AII.11. In Algebra I SOL A.9 (Technical Assistance Document – A.9), students study descriptive statistics through exploration of standard deviation, mean absolute deviation, and z-scores of data sets. Students compute these values for small defined populations, and the focus of instruction is on interpretation of these descriptive statistics. Algebra II, Objective 11 continues the study of descriptive statistics as students analyze properties of normal distributions and apply those properties to determine probabilities associated with areas under the standard normal curve. Students will be provided with the mean, standard deviation, and/or elements of samples or populations of normal distributions and asked to apply the properties of normal distributions to calculate probabilities associated with given elements of the data set.

Normal Distributions
Students examine many sets of data in their study of statistics. Some data have been provided to them and some data they have collected through surveys, experiments, or observations. Representation of (continued)
Normal Distribution and Z-scores (continued)

Teacher Notes and Elaborations (continued)

Data can take on many forms (line plots, stem-and-leaf plots, box-and-whisker plots, bar graphs, circle graphs, and histograms). Often teachers try to describe the physical shape of these representations in words or by using measures such as the arithmetic mean, the balance point of the data, or the median, the point at which the data is split into two equal amounts of data points. Another useful type of graphical representation is called a density curve, which models the pattern of a distribution. The distribution of certain types of data take on the appearance of a bell-shaped density curve called a normal curve. These collections of data are often naturally-occurring data or data produced by repetition in a mechanical process.

Examples of data that can be modeled by normal distributions include:
- heights of corn stalks in similar growing environments;
- heights of 16-year-old girls;
- blood pressures of 18-year-old males;
- weights of pennies in a given production year;
- lifespan of a specific electric motor; and
- standardized test scores like the ACT® or SAT®.

Standard Deviation

The standard deviation (\( \sigma \)) of a data set is a measure of the spread of data about the mean. The greater the value of the standard deviation, the more spread out the data are about the mean. The lesser (closer to 0) the value of the standard deviation, the closer the data are clustered about the mean.

Properties of normal distributions (normal probability distribution for continuous data)

Normal distributions are represented by a family of symmetric, bell-shaped curves called “normal” curves. Normal curves are defined by the mean and standard deviation of the data set. The arithmetic mean is located on the line of symmetry of the curve. In a normal distribution, the arithmetic mean is equivalent to the median and mode of the data set. Approximately 68 percent of the data values fall within one standard deviation (\( \sigma \)) of the mean (\( \mu \)), approximately 95 percent of the data values fall within two standard deviations of the mean, and approximately 99.7 percent of the data values fall within three standard deviations of the mean. This is often referred to as the 68-95-99.7 rule.

Figure 1 shows the approximate percentage of observations that fall within different partitions of the normal distribution.

![Figure 1](image_url)
Normal Distribution and Z-scores (continued)

A normal distribution with $\mu = 5$ and $\sigma = 1.5$ can be graphed on a graphing calculator. See the DOE Technical Assistance Document for Alg. II SOL AII.11 for calculator directions.

Determining probabilities associated with normal distributions

The cumulative probability of a specified range of values can be represented as the area under a normal distribution curve between the lower and upper bounds. When the range of values is an interval with lower and upper bounds equal to the mean or mean plus or minus one, two, or three standard deviations, the 68-95-99.7 rule can be used to determine probabilities. Graphing calculators can also be used to compute and graph the areas under normal curves.

Example 1

Given a normally distributed data set of 500 observations measuring tree heights in a forest, what is the approximate number of observations that fall within two standard deviations from the mean?

Solution

A quick sketch of the normal distribution will assist in solving this problem (refer to Figure 1). We know from the 68-95-99.7 rule that 95 percent of the data falls within two standard deviations from the mean. Therefore, approximately $500 \cdot 0.95 = 475$ of the trees’ height observations fall within two standard deviations from the mean.

Example 2

A normally distributed data set containing the number of ball bearings produced during a specified interval of time has a mean of 150 and a standard deviation of 10. What percentage of the observed values fall between 140 and 160?
Teacher Notes and Elaborations (continued)

Solution
From the sketch of a normally distributed data set, 140 is one standard deviation below the mean while 160 is one standard deviation above the mean. Therefore, approximately 34.1% + 34.1% = 68.2% of the data in this distribution falls between 140 and 160.

Example 3
Donna’s boss asked her to purchase a large number of 20-watt fluorescent light bulbs for their company. She has narrowed her search to two companies offering 20-watt bulbs for the same price.

The Bulb Emporium and Lights-R-Us each claim that the mean lifespan for their 20-watt bulbs is 10,000 hours. The lifespan of light bulbs has a distribution that is approximately normal. The Bulb Emporium’s distribution of the lifespan for 20-watt bulbs has a standard deviation of 1,000 hours and Lights-R-Us’ distribution of the lifespan of 20-watt bulbs has a standard deviation of 750 hours. Donna’s boss asked her to use probabilities associated with these normal distributions to make a purchasing decision.

Donna decided that she would compare the proportion of light bulbs from each company that would be expected to last for different intervals of time. She started with calculating the probability that a light bulb would be expected to last less than or equal to 9,000 hours. Letting $x$ represent the lifespan of a light bulb, $P(x \leq 9,000 \text{ hours})$ represents the probability that the lifespan of a light bulb would fall less than or equal to 9,000 hours in its normal distribution. Donna continued by finding $P(9,000 \leq x \leq 11,000 \text{ hours})$ and $P(x \geq 11,000 \text{ hours})$ for each company.

There are two ways to find the cumulative probability within a range of values on TI graphing calculators. The probability can be computed by finding the area under the curve bounded by a range of values using the ShadeNorm function, or it can be directly computed using the normalcdf function. On Casio graphing calculators, a feature similar to the TI’s ShadeNorm is only available in terms of the standard normal distribution using z-scores. The cumulative probabilities within a range of values can be computed on Casio graphing calculators. See the DOE Technical Assistance Document for Alg. II SOL AII.11 for calculator directions.

After recording the probabilities for the Bulb Emporium as shown in Figure 3, Donna calculated the same interval probabilities for Lights-R-Us and recorded them.

Figure 3

<table>
<thead>
<tr>
<th></th>
<th>Bulb Emporium ($\sigma = 1000$)</th>
<th>Lights-R-Us ($\sigma = 750$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(x \leq 9,000 \text{ hours})$</td>
<td>0.1587</td>
<td>0.0912</td>
</tr>
<tr>
<td>$P(9,000 \leq x \leq 11,000 \text{ hours})$</td>
<td>0.6827</td>
<td>0.8176</td>
</tr>
<tr>
<td>$P(x \geq 11,000 \text{ hours})$</td>
<td>0.1587</td>
<td>0.0912</td>
</tr>
</tbody>
</table>

In analyzing her data, Donna noticed that a lightbulb at Bulb Emporium had a higher probability of lasting less than 9,000 hours than a bulb at Lights-R-Us. However, a lightbulb at Bulb Emporium also had a higher probability of lasting longer than 11,000 hours than a bulb at Lights-R-Us. She determined that these two statistical

(continued)
Teacher Notes and Elaborations (continued)

probabilities offset one another. The middle interval of data showed that a bulb at Bulb Emporium had a lower probability of lasting between 9,000 and 11,000 hours, inclusively, than a bulb at Lights-R-Us. She therefore chose to purchase lightbulbs from Lights-R-Us.

z-Scores
A z-score, also called a standard score, is a measure of position derived from the mean and standard deviation of the data set. The z-score is a measure of how many standard deviations an element falls above or below the mean of the data set. The z-score has a positive value if the element lies above the mean and a negative value if the element lies below the mean.

A z-score associated with an element of a data set is calculated by subtracting the mean of the data set from the element and dividing the result by the standard deviation of the data set.

\[
z\text{-score } (z) = \frac{x - \mu}{\sigma},\]

where \(x\) represents an element of the data set, \(\mu\) represents the mean of the data set, and \(\sigma\) represents the standard deviation of the data set.

The standard normal curve is a normal distribution that has a mean of 0 and a standard deviation of 1. It is used to model z-scores obtained from normally distributed data. Prior to calculator technologies that can determine probabilities associated with normal distributions, the table of Standard Normal Probabilities, commonly referred to as a “z-table” was used to determine normal distribution probabilities.

Given a z-score \((z)\), the table of Standard Normal Probabilities shows the area under the curve to the left of \(z\). This area represents the proportion of observations with a z-score less than the one specified. Table rows show the z-score’s whole number and tenths place.

Table columns show the hundredths place. In the table of Standard Normal Probabilities provided for the state EOC SOL test in Algebra II, the cumulative probability from negative infinity to the z-score appears in table cells. Other tables of Standard Normal Probabilities show probabilities from the mean to the z-score.

Interpreting values from the table of Standard Normal Probabilities
A z-score associated with an element of a normal distribution is computed to be 1.23. The probability from the table of Standard Normal Probabilities associated with a z-score of 1.23 can be determined as indicated in Figure 4. The probability can be used differently based upon the context of the question.

- The probability that a data value will fall below the data value associated with a z-score of 1.23 is 0.8907 (89.07%).
- The data value associated with a z-score of 1.23 falls in the 89th percentile. This means that 89 percent of the data in the distribution fall below the value associated with a z-score of 1.23. (continued)
Normal Distribution and Z-scores (continued)

- The probability that a value from the data set will fall above this value is $1 - 0.8907 = 0.1093$ (10.93%).

Figure 4 shows the cumulative probability associated with a z-score of 1.23 using the table of standard normal probabilities.

From Example 3, Donna could have determined $P(x \leq 9,000\text{ hours})$ for Lights-R-Us using the table of Standard Normal Probabilities.

Given $\mu = 10,000$ and $\sigma = 750$, the z-score for 9,000 hours can be computed using the formula for z-score.

$$z = \frac{9000 - 10000}{750} = -1.33$$

In the table of Standard Normal Probabilities, $z = 1.33$ is associated with a cumulative probability of 0.0918, a value very close (difference of 0.0006) to the one found for Lights-R-Us in Figure 3, using a graphing calculator.

Example 4

In statistics, a percentile is the value of a variable below which a certain percent of observations fall. For example, the 20th percentile is the value (or score) below which 20 percent of the observations may be found. The term percentile and the related term percentile rank are often used in the reporting of scores from norm-referenced tests. The 25th percentile is also known as the first quartile ($Q_1$), the 50th percentile as the median or second quartile ($Q_2$), and the 75th percentile as the third quartile ($Q_3$).

The ACT® is an achievement test given nationally with normally distributed scores. Amy scored a 31 on the mathematics portion of her 2009 ACT®. The mean for the mathematics portion of the ACT® in 2009 was 21.0 and the standard deviation was 5.3. What percent of the population scored higher than Amy on the mathematics portion of the ACT®?

There are two “typical” approaches to find the percentage of the population that scored higher than Amy.
Teacher Notes and Resources (continued)

Teacher Notes and Resources (continued)

Normal Distribution and Z-scores (continued)

Solution A – Find the z-score and associated cumulative probability using the table of standard normal probabilities and subtract the cumulative probability from 1.

Calculate the z-score: 

\[ z = \frac{31 - 21.0}{5.3} = 1.89 \]

Look up the cumulative probability associated with a z-score of 1.89 on the table of standard normal probabilities. The probability of a test taker scoring a 31 (z-score = 1.89) is 0.9706 or 97 percent. This means that Amy scored in the 97th percentile and only 2.94 percent (1 – 0.9706 = 0.0294) scored higher than Amy.

Solution B – Find the probability of scoring higher than 31 using the graphing calculator.

The Texas Instruments (TI-83/84) graphing calculators can compute the percentage of the population scoring higher than 31 using the syntax `normalcdf(31,43,21.0, 5.3)`, given lower bound = 31, upper bound = 43, \( \mu = 21.0 \), and \( \sigma = 5.3 \). The resulting value is 0.0296 or 2.96 percent (beginning on page 58). Note: When choosing an upper bound (43 in this case), choose a number that will encompass all data values above the mean. The number 43 (rounded to 43 using \( \mu + 4 \sigma = 21+21.2 = 42.2 \)) is far enough above the mean to be a “safe” upper bound.

The Casio (9750/9850/9860) graphing calculators can compute the percentage of the population that score above a 31 by using the operation “Ncd” or “Normal C.D.” with a lower bound = 31, upper bound = 43, \( \sigma = 5.3 \), and \( \mu = 21.0 \). The calculated percentage of the population scoring above a 31 = 0.0296 or 2.96 percent. The Casio 9750/9860 can “DRAW” the representative area under the curve in terms of z-scores by pressing F6.

Example 4 extension

Amy took the ACT® and scored 31 on the mathematics portion of the test. Her friend Stephanie scored a 720 on the mathematics portion of her 2009 SAT®. Both the SAT® and the ACT® are achievement tests given nationally with scores that are normally distributed. The mean for the mathematics portion of the SAT® in 2009 was 515 and the standard deviation was 116. For the ACT®, the mean was 21 and the standard deviation was 5.3. Whose achievement was higher on the mathematics portion of their national achievement test?

Solution Since these two national achievement tests have different scoring scales, they cannot be compared directly. One way to compare them would be to find the cumulative probability (percentile) of each score using the associated z-score. Amy’s z-score is 1.89 (97th percentile) and Stephanie’s z-score is 1.77 (96th percentile). Therefore, Amy scored slightly higher than Stephanie on the mathematics portions of their respective national achievement tests.

Sample Instructional Strategies and Activities

Note: All exploration questions should be in the real-world context of normally distributed data sets.

Given a normally distributed data set with a specified mean and standard deviation, explain how the number of values expected to be above or below a certain value can be determined.

(continued)
<table>
<thead>
<tr>
<th>Sample Instructional Strategies and Activities (continued)</th>
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<tbody>
<tr>
<td>• Given normally distributed data, explain how you can determine how many values or what percentage of values are expected to fall within one, two or three standard deviations of the mean.</td>
</tr>
<tr>
<td>• Compare and contrast graphs of normal distributions that have the same mean but different standard deviations or different means and the same standard deviation.</td>
</tr>
<tr>
<td>• Given a normally distributed data set with a specified mean and standard deviation, explain how to determine the probability and/or area under the curve for</td>
</tr>
<tr>
<td>○ an element that has a value greater than a given value;</td>
</tr>
<tr>
<td>○ an element that has a value less than a given value; or</td>
</tr>
<tr>
<td>○ an element that has a value between two given values.</td>
</tr>
<tr>
<td>• Given the mean and standard deviation of two different normally distributed data sets, and a value from each data set, compare the values using their corresponding z-scores and percentiles.</td>
</tr>
<tr>
<td>• Given normally distributed data with specified mean and standard deviation, determine the probability that a randomly selected value will have a z-score within a certain range of values.</td>
</tr>
</tbody>
</table>
### Other SOL Topics – Variation, Curve of Best Fit

#### Strand: Statistics

**SOL AII.10**
The student will identify, create, and solve real-world problems involving inverse variation, joint variation, and a combination of direct and inverse variations.

**SOL AII.9**
The student will collect and analyze data, determine the equation of the curve of best fit, make predictions, and solve real-world problems, using mathematical models. Mathematical models will include polynomial, exponential, and logarithmic functions.

#### Essential Knowledge and Skills

<table>
<thead>
<tr>
<th>The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Translate “y varies jointly as x and z” as $y = kxz$.</td>
</tr>
<tr>
<td>• Translate “y is directly proportional to x” as $y = kx$.</td>
</tr>
<tr>
<td>• Translate “y is inversely proportional to x” as $y = \frac{k}{x}$.</td>
</tr>
<tr>
<td>• Given a situation, determine the value of the constant of proportionality.</td>
</tr>
<tr>
<td>• Set up and solve problems, including real-world problems, involving inverse variation, joint variation, and a combination of direct and inverse variations.</td>
</tr>
<tr>
<td>• Collect and analyze data.</td>
</tr>
<tr>
<td>• Investigate scatterplots to determine if patterns exist and then identify the patterns.</td>
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<tr>
<td>• Find an equation for the curve of best fit for data, using a graphing calculator. Models will include polynomial, exponential, and logarithmic functions.</td>
</tr>
<tr>
<td>• Make predictions, using data, scatterplots, or the equation of the curve of best fit.</td>
</tr>
<tr>
<td>• Given a set of data, determine the model that would best describe the data.</td>
</tr>
</tbody>
</table>

#### Essential Understandings

- Real-world problems can be modeled and solved by using inverse variation, joint variation, and a combination of direct and inverse variations.
- Joint variation is a combination of direct variations.
- Data and scatterplots may indicate patterns that can be modeled with an algebraic equation.
- Graphing calculators can be used to collect, organize, picture, and create an algebraic model of the data.
- Data that fit polynomial $(f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0)$, where $n$ is a nonnegative integer, and the coefficients are real numbers, exponential ($y = b^x$), and logarithmic ($y = \log_b x$) models arise from real-world situations.

(continued)
# Other SOL Topics – Variation

<table>
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<th>Resources</th>
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<td>2-3 Direct Variation</td>
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<tr>
<td>9-1 Inverse Variation</td>
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<td>9-2 Graphing Inverse Variation</td>
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<tr>
<td><strong>HCPS Algebra 2 Online!</strong></td>
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<tr>
<td>Variation</td>
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<tr>
<td><strong>DOE ESS Lesson Plans:</strong></td>
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<tr>
<td>• Types of Variation (PDF) (Word)</td>
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</table>

<table>
<thead>
<tr>
<th>Key Vocabulary</th>
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<tr>
<td>combined variation</td>
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<tr>
<td>inverse variation</td>
</tr>
<tr>
<td>constant of proportionality</td>
</tr>
<tr>
<td>joint variation</td>
</tr>
<tr>
<td>proportional</td>
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</table>

**Essential Questions**

• What is the difference between direct and inverse variation?
• What is joint variation?
• What is combined variation?

**Teacher Notes and Elaborations**

Two variable quantities are *proportional* if one of them is always the product of the other and a constant quantity.

Rational equations, to include direct and inverse variation, can be solved algebraically.

*Direct variation:*

When two variables are related so that their ratio remains constant, one of the variables is said to vary directly to the other variables or the variables are said to vary proportionately. A linear function defined by an equation of the form \( y = kx \), where \( k \neq 0 \), represents direct variation. The constant of variation (*constant of proportionality*) is \( k \). For a linear function to be a direct variation the graph must be a non-horizontal line through the origin.

*Inverse variation:*

When the ratio of one variable is constant to the reciprocal of the other variable, one of the variables is said to vary inversely to the other variable. A function of the form \( y = \frac{k}{x} \) or \( xy = k \), where \( k \neq 0 \), is an inverse variation. The constant of variation (*constant of proportionality*) is \( k \).

*Joint variation:*

If \( y = kxz \) then \( y \) varies jointly as \( x \) and \( z \), and the constant of variation is \( k \). In joint variation one quantity varies directly as two quantities. This is a combination of direct variations. For example: In a rectangular prism the equation for volume is \( V = lwh \). If \( h = 8 \) then the volume varies jointly as the length \( (l) \) and width \( (w) \). The constant of variation is 8.

*Combined variation:*

This involves both direct and inverse variations occurring in the same equation.

Practical problems can be modeled and solved by using direct and/or inverse variations.

**Sample Instructional Strategies and Activities**

• In groups have students develop problems that involve the different variations. Students explain how they know what type of variation this problem represents.
### Other SOL Topics – Curve of Best Fit

**Resources**

**Textbook:**
- 2-4 Using Linear Models
- 5-1 Modeling Data with Quadratic Functions
- pg. 240 Modeling Using Residuals Technology Extension
- 6-1 Polynomial Functions
- pg. 430 Fitting Exponential Curves to Data Technology Extension

**HCPS Algebra 2 Online!**
- Curve of Best Fit

**DOE ESS Lesson Plans:**
- Curve of Best Fit (PDF) (Word)

<table>
<thead>
<tr>
<th>Key Vocabulary</th>
</tr>
</thead>
<tbody>
<tr>
<td>curve of best fit</td>
</tr>
<tr>
<td>mathematical model</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Essential Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>How do various algebraic equations fit real world data?</td>
</tr>
<tr>
<td>How can the curve-of-best-fit help predict trends of data?</td>
</tr>
<tr>
<td>How are the equations-of-best-fit determined on a graphing calculator?</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Teacher Notes and Elaborations</th>
</tr>
</thead>
<tbody>
<tr>
<td>A <em>mathematical model</em> usually describes a system by a set of variables and a set of equations that establish relationships between the variables.</td>
</tr>
</tbody>
</table>

Data that fit polynomial, exponential, and logarithmic models arise from practical situations. The analyzing of data to determine a curve of best fit has numerous real-world applications such as oceanography, business, economics, and agriculture.

*scatterplot* visually shows the nature of a relationship and both its shape and dispersion. A *curve of best fit* may be drawn to show the approximate relationship formed by the plotted points.

The use of scatterplots on a graphing calculator will determine if the relationship is polynomial, exponential, or logarithmic.

Graphing calculators can be used to collect, organize, picture and create an algebraic model of the data.

<table>
<thead>
<tr>
<th>Sample Instructional Strategies and Activities</th>
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</thead>
<tbody>
<tr>
<td>Collect information from students on how many miles they drove that week and how many gallons of gasoline they used. Draw a scatterplot and make a prediction about how much gasoline they will use each year. Next, students will estimate the amount of money they will spend on gas this year, if gasoline cost $3.89 per gallon.</td>
</tr>
<tr>
<td>Students try to determine how tall a person is whose femur is 17 inches long. They measure their own femurs and their heights and enter the data into a graphing calculator or computer to get a scatterplot. Find the equation for the curve of best fit and use it to make predictions.</td>
</tr>
<tr>
<td>Strand: Statistics</td>
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<tr>
<td>---------------------------------------------</td>
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<tr>
<td>SOL AII.12</td>
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</table>
### Permutations and Combinations (continued)

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<th>Resources</th>
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<tr>
<td>HCPS Algebra 2 Online!:</td>
<td><strong>Permutations and Combinations</strong></td>
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<tr>
<td>DOE ESS Lesson Plans:</td>
<td>- Combinations and Permutations (PDF) (Word)</td>
</tr>
</tbody>
</table>

#### Key Vocabulary
- combination
- permutation
- factorial

#### Essential Questions
- What is a permutation and how is it determined?
- What is a combination and how is it determined?
- What is the difference between a permutation and a combination of the same items?
- When is a permutation or a combination used?

#### Teacher Notes and Elaborations

A *factorial* is the product of all the positive integers through the given integer (e.g., $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$).

A *permutation* is an arrangement of items in a particular order. It can often be found by using the Fundamental Counting Principle or factorial notation. The number of permutation of $n$ objects is given as $n!$.

The permutations of $n$ objects taken $r$ at a time is $P(n, r)$ or $nP_r = \frac{n!}{(n-r)!}$ for $0 \leq r \leq n$.

A *combination* is a selection in which order does not matter. The number of combinations of $n$ items taken $r$ at a time is $C(n, r)$ or $nC_r = \frac{n!}{r!(n-r)!}$ for $0 \leq r \leq n$.

The formula for combinations is like the formula for permutations except that it contains the factor $r!$ to compensate for duplicate combinations.

**Combination rule:**
The number of combinations of $r$ items selected from $n$ different items is: $nC_r = \frac{n!}{r!(n-r)!}$ The following conditions must apply: a total of $n$ different items available, select $r$ of the $n$ items without replacement (consider rearrangements of the same items to be the same – the combination of ABC is the same as CBA).

**Example:** A Board of Trustees at the XYZ Company has 9 members. Each year, they elect a 3-person committee to oversee buildings and grounds. Each year, they also elect a chairperson, vice chairperson, and secretary. When the board elects the buildings and grounds committee, how many different 3-person committees are possible?

**Solution:** Because order is irrelevant when electing the buildings and grounds committee, the number of combinations of $r = 3$ people selected from the $n = 9$ available people is

(continued)
**Teacher Notes and Elaborations (continued)**

For calculating combinations, we use the formula:

\[ _nC_r = \frac{n!}{(n-r)!r!} \]

For the given example with \( n = 9 \) available people and \( r = 3 \) people selected, we have:

\[ _9C_3 = \frac{9!}{(9-3)!3!} = 84 \]

Because order does matter with the slates of candidates, the number of sequences (or permutations) of \( r = 3 \) people selected from the \( n = 9 \) available people is:

\[ _nP_r = \frac{n!}{(n-r)!} = \frac{9!}{(9-3)!} = 504 \]

There are 84 different possible committees of 3 board members, but there are 504 different possible slates of candidates.

**Permutation Rule (When Some Items Are Identical to Others)**

If there are \( n \) items with \( n_1 \) alike, \( n_2 \) alike,...\( n_k \) alike, the number of permutations of all \( n \) items is:

\[ \frac{n!}{n_1!n_2!...n_k!} \]

**Example:** Consider the letters BBBBBAAAA, which represent a sequence of recent years in which the Dow Jones Industrial Average was below \( B \) the mean or above \( A \) the mean. How many ways can the letters BBBBBAAAA be arranged? Does it appear that the sequence is random? Is there a pattern suggesting that it would be wise to invest in stocks?

**Solution:** In the sequence BBBBBAAAA, \( n = 9 \) items, with \( n_1 = 5 \) alike and \( n_2 = 4 \) others that are alike. The number of permutations is computed as follows:

\[ \frac{9!}{5!4!} = 126 \]

There are 126 different ways that the letters can be arranged. Because there are 126 different possible arrangements and only two of them (BBBBBAAAA and AAAABBBBB) result in the letters all grouped together, it appears that the sequence is not random. Because all of the below values occur at the beginning and all of the above values occur at the end, it suggests that it would be wise to invest in stocks.

A permutation problem occurs when different orderings of the same items are counted separately. A combination problem occurs when different orderings of the same items are not counted separately.

**Sample Instructional Strategies and Activities**

**Skittles Lab:**

1. How many skittles are in the bag?
2. List the total for each color of skittles in a bag.
   - Answer the following questions. (Students need to be thorough in their explanations.)
     a. How many ways are there to select 5 skittles from your bag, disregarding the order of selection?
     b. In a bag of \( x \) skittles how many ways can a red, green, and yellow be selected?
     c. How many permutations can be made for red skittles in your sample?
     d. How many different arrangements can be made for red, yellow, green, purple and orange?
     e. How many different way are there to pick a yellow from your bag?
Algebra II Formula Sheet
2009 Mathematics Standards of Learning

Geometric Formulas:

\[ A = \frac{1}{2}bh \]
\[ p = 4s \]
\[ A = s^2 \]
\[ p = 2l + 2w \]
\[ A = lw \]
\[ a^2 + b^2 = c^2 \]

Quadratic Formula:

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \], where \( ax^2 + bx + c = 0 \) and \( a \neq 0 \)

Statistics Formula:

Given:
- \( x \) represents an element of the data set,
- \( \mu \) represents the mean of the data set, and
- \( \sigma \) represents the standard deviation of the data set

\[ z \text{-score} (z) = \frac{x - \mu}{\sigma} \]

Sequence and Series Formulas:

Given:
- \( a_n \) represents the value of \( n \)th term
- \( S_n \) represents the sum of first \( n \) terms
- \( S_\infty \) represents the sum of an infinite geometric series
- \( r \) represents the common ratio
- \( d \) represents the common difference

Arithmetic

\[ a_n = a_1 + (n-1)d \]
\[ a_n = a_{n-1} + d \]
\[ S_n = \frac{n}{2}(a_1 + a_n) \]
\[ S_n = \frac{n}{2}[2a_1 + (n-1)d] \]

Geometric

\[ a_n = a_1r^{n-1} \]
\[ a_n = a_{n-1} \cdot r \]
\[ S_n = \frac{a_1(1-r^n)}{(1-r)} \], \( r \neq 1 \)
\[ S_\infty = \frac{a_1}{(1-r)}, \quad |r| < 1 \]
Mathematics Standards of Learning

Curriculum Framework 2009

Algebra II

Board of Education
Commonwealth of Virginia
Henrico Curriculum Framework

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Introduction

The 2009 *Mathematics Standards of Learning* Curriculum Framework is a companion document to the 2009 *Mathematics Standards of Learning* and amplifies the *Mathematics Standards of Learning* by defining the content knowledge, skills, and understandings that are measured by the Standards of Learning assessments. The Curriculum Framework provides additional guidance to school divisions and their teachers as they develop an instructional program appropriate for their students. It assists teachers in their lesson planning by identifying essential understandings, defining essential content knowledge, and describing the intellectual skills students need to use. This supplemental framework delineates in greater specificity the content that all teachers should teach and all students should learn.

Each topic in the *Mathematics Standards of Learning* Curriculum Framework is developed around the Standards of Learning. The format of the Curriculum Framework facilitates teacher planning by identifying the key concepts, knowledge and skills that should be the focus of instruction for each standard. The Curriculum Framework is divided into two columns: Essential Understandings and Essential Knowledge and Skills. The purpose of each column is explained below.

**Essential Understandings**
This section delineates the key concepts, ideas and mathematical relationships that all students should grasp to demonstrate an understanding of the Standards of Learning.

**Essential Knowledge and Skills**
Each standard is expanded in the Essential Knowledge and Skills column. What each student should know and be able to do in each standard is outlined. This is not meant to be an exhaustive list nor a list that limits what is taught in the classroom. It is meant to be the key knowledge and skills that define the standard.

The Curriculum Framework serves as a guide for Standards of Learning assessment development. Assessment items may not and should not be a verbatim reflection of the information presented in the Curriculum Framework. Students are expected to continue to apply knowledge and skills from Standards of Learning presented in previous grades as they build mathematical expertise.
### TOPIC: EXPRESSIONS AND OPERATIONS

#### ALGEBRA II

**STANDARD AII.1**

The student, given rational, radical, or polynomial expressions, will

- a) add, subtract, multiply, divide, and simplify rational algebraic expressions;
- b) add, subtract, multiply, divide, and simplify radical expressions containing rational numbers and variables, and expressions containing rational exponents;
- c) write radical expressions as expressions containing rational exponents and vice versa; and
- d) factor polynomials completely.

### ESSENTIAL UNDERSTANDINGS

- Computational skills applicable to numerical fractions also apply to rational expressions involving variables.
- Radical expressions can be written and simplified using rational exponents.
- Only radicals with a common radicand and index can be added or subtracted.
- A relationship exists among arithmetic complex fractions, algebraic complex fractions, and rational numbers.
- The complete factorization of polynomials has occurred when each factor is a prime polynomial.
- Pattern recognition can be used to determine complete factorization of a polynomial.

### ESSENTIAL KNOWLEDGE AND SKILLS

The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to

- Add, subtract, multiply, and divide rational algebraic expressions.
- Simplify a rational algebraic expression with common monomial or binomial factors.
- Recognize a complex algebraic fraction, and simplify it as a quotient or product of simple algebraic fractions.
- Simplify radical expressions containing positive rational numbers and variables.
- Convert from radical notation to exponential notation, and vice versa.
- Add and subtract radical expressions.
- Multiply and divide radical expressions not requiring rationalizing the denominators.
ALGEBRA II  
STANDARD AII.1  
The student, given rational, radical, or polynomial expressions, will  
\(a\) add, subtract, multiply, divide, and simplify rational algebraic expressions;  
\(b\) add, subtract, multiply, divide, and simplify radical expressions containing rational numbers and variables, and expressions containing rational exponents;  
\(c\) write radical expressions as expressions containing rational exponents and vice versa; and  
\(d\) factor polynomials completely.

<table>
<thead>
<tr>
<th>ESSENTIAL UNDERSTANDINGS</th>
<th>ESSENTIAL KNOWLEDGE AND SKILLS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>• Factor polynomials by applying general patterns including difference of squares, sum and difference of cubes, and perfect square trinomials.</td>
</tr>
<tr>
<td></td>
<td>• Factor polynomials completely over the integers.</td>
</tr>
<tr>
<td></td>
<td>• Verify polynomial identities including the difference of squares, sum and difference of cubes, and perfect square trinomials.†</td>
</tr>
</tbody>
</table>

†Revised March 2011
**ALGEBRA II**  
**STANDARD AII.2**

The student will investigate and apply the properties of arithmetic and geometric sequences and series to solve real-world problems, including writing the first $n$ terms, finding the $n^{th}$ term, and evaluating summation formulas. Notation will include $\sum$ and $a_n$.

<table>
<thead>
<tr>
<th>ESSENTIAL UNDERSTANDINGS</th>
<th>ESSENTIAL KNOWLEDGE AND SKILLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Sequences and series arise from real-world situations.</td>
<td>The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to</td>
</tr>
<tr>
<td>• The study of sequences and series is an application of the investigation of patterns.</td>
<td>• Distinguish between a sequence and a series.</td>
</tr>
<tr>
<td>• A sequence is a function whose domain is the set of natural numbers.</td>
<td>• Generalize patterns in a sequence using explicit and recursive formulas.</td>
</tr>
<tr>
<td>• Sequences can be defined explicitly and recursively.</td>
<td>• Use and interpret the notations $\sum$, $n$, $n^{th}$ term, and $a_n$.</td>
</tr>
<tr>
<td></td>
<td>• Given the formula, find $a_n$ (the $n^{th}$ term) for an arithmetic or a geometric sequence.</td>
</tr>
<tr>
<td></td>
<td>• Given formulas, write the first $n$ terms and find the sum, $S_n$, of the first $n$ terms of an arithmetic or geometric series.</td>
</tr>
<tr>
<td></td>
<td>• Given the formula, find the sum of a convergent infinite series.</td>
</tr>
<tr>
<td></td>
<td>• Model real-world situations using sequences and series.</td>
</tr>
</tbody>
</table>
ALGEBRA II
STANDARD AII.3

The student will perform operations on complex numbers, express the results in simplest form using patterns of the powers of $i$, and identify field properties that are valid for the complex numbers.

<table>
<thead>
<tr>
<th>ESSENTIAL UNDERSTANDINGS</th>
<th>ESSENTIAL KNOWLEDGE AND SKILLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Complex numbers are organized into a hierarchy of subsets.</td>
<td>The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to</td>
</tr>
<tr>
<td>• A complex number multiplied by its conjugate is a real number.</td>
<td>• Recognize that the square root of $-1$ is represented as $i$.</td>
</tr>
<tr>
<td>• Equations having no real number solutions may have solutions in the set of complex numbers.</td>
<td>• Determine which field properties apply to the complex number system.</td>
</tr>
<tr>
<td>• Field properties apply to complex numbers as well as real numbers.</td>
<td>• Simplify radical expressions containing negative rational numbers and express in $a+bi$ form.</td>
</tr>
<tr>
<td>• All complex numbers can be written in the form $a+bi$ where $a$ and $b$ are real numbers and $i$ is $\sqrt{-1}$.</td>
<td>• Simplify powers of $i$.</td>
</tr>
<tr>
<td></td>
<td>• Add, subtract, and multiply complex numbers.</td>
</tr>
<tr>
<td></td>
<td>• Place the following sets of numbers in a hierarchy of subsets: complex, pure imaginary, real, rational, irrational, integers, whole, and natural.</td>
</tr>
<tr>
<td></td>
<td>• Write a real number in $a+bi$ form.</td>
</tr>
<tr>
<td></td>
<td>• Write a pure imaginary number in $a+bi$ form.</td>
</tr>
</tbody>
</table>
**ALGEBRA II**  
**STANDARD AII.4**

The student will solve, algebraically and graphically,
- a) absolute value equations and inequalities;
- b) quadratic equations over the set of complex numbers;
- c) equations containing rational algebraic expressions; and
- d) equations containing radical expressions.

Graphing calculators will be used for solving and for confirming the algebraic solutions.

<table>
<thead>
<tr>
<th>ESSENTIAL UNDERSTANDINGS</th>
<th>ESSENTIAL KNOWLEDGE AND SKILLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>• A quadratic function whose graph does not intersect the (x)-axis has roots with imaginary components.</td>
<td>The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to</td>
</tr>
<tr>
<td>• The quadratic formula can be used to solve any quadratic equation.</td>
<td>• Solve absolute value equations and inequalities algebraically and graphically.</td>
</tr>
<tr>
<td>• The value of the discriminant of a quadratic equation can be used to describe the number of real and complex solutions.</td>
<td>• Solve a quadratic equation over the set of complex numbers using an appropriate strategy.</td>
</tr>
<tr>
<td>• The definition of absolute value (for any real numbers (a) and (b), where (b \geq 0), if (</td>
<td>d</td>
</tr>
<tr>
<td>• Absolute value inequalities can be solved graphically or by using a compound statement.</td>
<td>• Solve equations containing rational algebraic expressions with monomial or binomial denominators algebraically and graphically.</td>
</tr>
<tr>
<td>• Real-world problems can be interpreted, represented, and solved using equations and inequalities.</td>
<td>• Solve an equation containing a radical expression algebraically and graphically.</td>
</tr>
<tr>
<td>• The process of solving radical or rational equations can lead to extraneous solutions.</td>
<td>• Verify possible solutions to an equation containing rational or radical expressions.</td>
</tr>
</tbody>
</table>
## ALGEBRA II
### STANDARD AII.4

The student will solve, algebraically and graphically,

a) absolute value equations and inequalities;

b) quadratic equations over the set of complex numbers;

c) equations containing rational algebraic expressions; and

d) equations containing radical expressions.

Graphing calculators will be used for solving and for confirming the algebraic solutions.

### ESSENTIAL UNDERSTANDINGS

- Equations can be solved in a variety of ways.
- Set builder notation may be used to represent solution sets of equations and inequalities.

### ESSENTIAL KNOWLEDGE AND SKILLS

- Apply an appropriate equation to solve a real-world problem.
- Recognize that the quadratic formula can be derived by applying the completion of squares to any quadratic equation in standard form.†

†Revised March 2011
### ALGEBRA II

**STANDARD AII.5**

The student will solve nonlinear systems of equations, including linear-quadratic and quadratic-quadratic, algebraically and graphically. Graphing calculators will be used as a tool to visualize graphs and predict the number of solutions.

<table>
<thead>
<tr>
<th>ESSENTIAL UNDERSTANDINGS</th>
<th>ESSENTIAL KNOWLEDGE AND SKILLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Solutions of a nonlinear system of equations are numerical values that satisfy every equation in the system.</td>
<td>The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to</td>
</tr>
<tr>
<td>• The coordinates of points of intersection in any system of equations are solutions to the system.</td>
<td></td>
</tr>
<tr>
<td>• Real-world problems can be interpreted, represented, and solved using systems of equations.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
ALGEBRA II
STANDARD AII.6

The student will recognize the general shape of function (absolute value, square root, cube root, rational, polynomial, exponential, and logarithmic) families and will convert between graphic and symbolic forms of functions. A transformational approach to graphing will be employed. Graphing calculators will be used as a tool to investigate the shapes and behaviors of these functions.

<table>
<thead>
<tr>
<th>ESSENTIAL UNDERSTANDINGS</th>
<th>ESSENTIAL KNOWLEDGE AND SKILLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>• The graphs/equations for a family of functions can be determined using a transformational approach.</td>
<td>The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to</td>
</tr>
<tr>
<td>• Transformations of graphs include translations, reflections, and dilations.</td>
<td>• Recognize graphs of parent functions.</td>
</tr>
<tr>
<td>• A parent graph is an anchor graph from which other graphs are derived with transformations.</td>
<td>• Given a transformation of a parent function, identify the graph of the transformed function.</td>
</tr>
<tr>
<td></td>
<td>• Given the equation and using a transformational approach, graph a function.</td>
</tr>
<tr>
<td></td>
<td>• Given the graph of a function, identify the parent function.</td>
</tr>
<tr>
<td></td>
<td>• Given the graph of a function, identify the transformations that map the preimage to the image in order to determine the equation of the image.</td>
</tr>
<tr>
<td></td>
<td>• Using a transformational approach, write the equation of a function given its graph.</td>
</tr>
</tbody>
</table>
**TOPIC: FUNCTIONS**

**ALGEBRA II
STANDARD AII.7**

The student will investigate and analyze functions algebraically and graphically. Key concepts include

- a) domain and range, including limited and discontinuous domains and ranges;
- b) zeros;
- c) \(x\)- and \(y\)-intercepts;
- d) intervals in which a function is increasing or decreasing;
- e) asymptotes;
- f) end behavior;
- g) inverse of a function; and
- h) composition of multiple functions.

Graphing calculators will be used as a tool to assist in investigation of functions.

### ESSENTIAL UNDERSTANDINGS

- Functions may be used to model real-world situations.
- The domain and range of a function may be restricted algebraically or by the real-world situation modeled by the function.
- A function can be described on an interval as increasing, decreasing, or constant.
- Asymptotes may describe both local and global behavior of functions.
- End behavior describes a function as \(x\) approaches positive and negative infinity.
- A zero of a function is a value of \(x\) that makes \(f(x)\) equal zero.

### ESSENTIAL KNOWLEDGE AND SKILLS

The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to

- Identify the domain, range, zeros, and intercepts of a function presented algebraically or graphically.
- Describe restricted/discontinuous domains and ranges.
- Given the graph of a function, identify intervals on which the function is increasing and decreasing.
- Find the equations of vertical and horizontal asymptotes of functions.
- Describe the end behavior of a function.
- Find the inverse of a function.
- Graph the inverse of a function as a reflection across the line.
TOPIC: FUNCTIONS

ALGEBRA II
STANDARD AII.7

The student will investigate and analyze functions algebraically and graphically. Key concepts include

a) domain and range, including limited and discontinuous domains and ranges;
b) zeros;
c) \(x\)- and \(y\)-intercepts;
d) intervals in which a function is increasing or decreasing;
e) asymptotes;
f) end behavior;
g) inverse of a function; and
h) composition of multiple functions.

Graphing calculators will be used as a tool to assist in investigation of functions.

<table>
<thead>
<tr>
<th>ESSENTIAL UNDERSTANDINGS</th>
<th>ESSENTIAL KNOWLEDGE AND SKILLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>• If ((a, b)) is an element of a function, then ((b, a)) is an element of the inverse of the function.</td>
<td>• (y = x).</td>
</tr>
<tr>
<td>• Exponential ((y = a^x)) and logarithmic ((y = \log_a x)) functions are inverses of each other.</td>
<td>• Investigate exponential and logarithmic functions, using the graphing calculator.</td>
</tr>
<tr>
<td>• Functions can be combined using composition of functions.</td>
<td>• Convert between logarithmic and exponential forms of an equation with bases consisting of natural numbers.</td>
</tr>
<tr>
<td></td>
<td>• Find the composition of two functions.</td>
</tr>
<tr>
<td></td>
<td>• Use composition of functions to verify two functions are inverses.</td>
</tr>
</tbody>
</table>
### TOPIC: FUNCTIONS

#### ALGEBRA II

**STANDARD AII.8**

The student will investigate and describe the relationships among solutions of an equation, zeros of a function, \( x \)-intercepts of a graph, and factors of a polynomial expression.

<table>
<thead>
<tr>
<th>ESSENTIAL UNDERSTANDINGS</th>
<th>ESSENTIAL KNOWLEDGE AND SKILLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>- The <em>Fundamental Theorem of Algebra</em> states that, including complex and repeated solutions, an ( n )th degree polynomial equation has exactly ( n ) roots (solutions).</td>
<td></td>
</tr>
<tr>
<td>- The following statements are equivalent:</td>
<td></td>
</tr>
<tr>
<td>- ( k ) is a zero of the polynomial function ( f );</td>
<td></td>
</tr>
<tr>
<td>- ( (x - k) ) is a factor of ( f(x) );</td>
<td></td>
</tr>
<tr>
<td>- ( k ) is a solution of the polynomial equation ( f(x) = 0 ); and</td>
<td></td>
</tr>
<tr>
<td>- ( k ) is an ( x )-intercept for the graph of ( y = f(x) ).</td>
<td></td>
</tr>
<tr>
<td>The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to</td>
<td></td>
</tr>
<tr>
<td>- Describe the relationships among solutions of an equation, zeros of a function, ( x )-intercepts of a graph, and factors of a polynomial expression.</td>
<td></td>
</tr>
<tr>
<td>- Define a polynomial function, given its zeros.</td>
<td></td>
</tr>
<tr>
<td>- Determine a factored form of a polynomial expression from the ( x )-intercepts of the graph of its corresponding function.</td>
<td></td>
</tr>
<tr>
<td>- For a function, identify zeros of multiplicity greater than 1 and describe the effect of those zeros on the graph of the function.</td>
<td></td>
</tr>
<tr>
<td>- Given a polynomial equation, determine the number of real solutions and nonreal solutions.</td>
<td></td>
</tr>
</tbody>
</table>
ALGEBRA II
STANDARD AII.9

The student will collect and analyze data, determine the equation of the curve of best fit, make predictions, and solve real-world problems, using mathematical models. Mathematical models will include polynomial, exponential, and logarithmic functions.

<table>
<thead>
<tr>
<th>ESSENTIAL UNDERSTANDINGS</th>
<th>ESSENTIAL KNOWLEDGE AND SKILLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Data and scatterplots may indicate patterns that can be modeled with an algebraic equation.</td>
<td>The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to</td>
</tr>
<tr>
<td>• Graphing calculators can be used to collect, organize, picture, and create an algebraic model of the data.</td>
<td>• Collect and analyze data.</td>
</tr>
<tr>
<td>• Data that fit polynomial ( f(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 ); where ( n ) is a nonnegative integer, and the coefficients are real numbers), exponential ( y = b^x ), and logarithmic ( y = \log_b x ) models arise from real-world situations.</td>
<td>• Investigate scatterplots to determine if patterns exist and then identify the patterns.</td>
</tr>
<tr>
<td></td>
<td>• Find an equation for the curve of best fit for data, using a graphing calculator. Models will include polynomial, exponential, and logarithmic functions.</td>
</tr>
<tr>
<td></td>
<td>• Make predictions, using data, scatterplots, or the equation of the curve of best fit.</td>
</tr>
<tr>
<td></td>
<td>• Given a set of data, determine the model that would best describe the data.</td>
</tr>
</tbody>
</table>
ALGEBRA II
STANDARD AII.10
The student will identify, create, and solve real-world problems involving inverse variation, joint variation, and a combination of direct and inverse variations.

<table>
<thead>
<tr>
<th>ESSENTIAL UNDERSTANDINGS</th>
<th>ESSENTIAL KNOWLEDGE AND SKILLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Real-world problems can be modeled and solved by using inverse variation, joint variation, and a combination of direct and inverse variations.</td>
<td></td>
</tr>
<tr>
<td>• Joint variation is a combination of direct variations.</td>
<td></td>
</tr>
<tr>
<td>The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to</td>
<td></td>
</tr>
<tr>
<td>• Translate “y varies jointly as x and z” as ( y = kxz ).</td>
<td></td>
</tr>
<tr>
<td>• Translate “y is directly proportional to x” as ( y = kx ).</td>
<td></td>
</tr>
<tr>
<td>• Translate “y is inversely proportional to x” as ( y = \frac{k}{x} ).</td>
<td></td>
</tr>
<tr>
<td>• Given a situation, determine the value of the constant of proportionality.</td>
<td></td>
</tr>
<tr>
<td>• Set up and solve problems, including real-world problems, involving inverse variation, joint variation, and a combination of direct and inverse variations.</td>
<td></td>
</tr>
</tbody>
</table>
**ALGEBRA II**  
**STANDARD AII.11**

The student will identify properties of a normal distribution and apply those properties to determine probabilities associated with areas under the standard normal curve.

<table>
<thead>
<tr>
<th>ESSENTIAL UNDERSTANDINGS</th>
<th>ESSENTIAL KNOWLEDGE AND SKILLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>• A normal distribution curve is a symmetrical, bell-shaped curve defined by the mean and the standard deviation of a data set. The mean is located on the line of symmetry of the curve.</td>
<td>The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to identify the properties of a normal probability distribution.</td>
</tr>
<tr>
<td>• Areas under the curve represent probabilities associated with continuous distributions.</td>
<td>• Identify the properties of a normal probability distribution.</td>
</tr>
<tr>
<td>• The normal curve is a probability distribution and the total area under the curve is 1.</td>
<td>• Describe how the standard deviation and the mean affect the graph of the normal distribution.</td>
</tr>
<tr>
<td>• For a normal distribution, approximately 68 percent of the data fall within one standard deviation of the mean, approximately 95 percent of the data fall within two standard deviations of the mean, and approximately 99.7 percent of the data fall within three standard deviations of the mean.</td>
<td>• Compare two sets of normally distributed data using a standard normal distribution and z-scores.</td>
</tr>
<tr>
<td>• The mean of the data in a standard normal distribution is 0 and the standard deviation is 1.</td>
<td>• Represent probability as area under the curve of a standard normal probability distribution.</td>
</tr>
<tr>
<td>• The standard normal curve allows for the comparison of data from different normal distributions.</td>
<td>• Use the graphing calculator or a standard normal probability table to determine probabilities or percentiles based on z-scores.</td>
</tr>
<tr>
<td>• A z-score is a measure of position derived from the mean and standard deviation of data.</td>
<td></td>
</tr>
</tbody>
</table>
ALGEBRA II
STANDARD AII.11
The student will identify properties of a normal distribution and apply those properties to determine probabilities associated with areas under the standard normal curve.

<table>
<thead>
<tr>
<th>ESSENTIAL UNDERSTANDINGS</th>
<th>ESSENTIAL KNOWLEDGE AND SKILLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>• A z-score expresses, in standard deviation units, how far an element falls from the mean of the data set.</td>
<td></td>
</tr>
<tr>
<td>• A z-score is a derived score from a given normal distribution.</td>
<td></td>
</tr>
<tr>
<td>• A standard normal distribution is the set of all z-scores.</td>
<td></td>
</tr>
</tbody>
</table>
ALGEBRA II  
STANDARD AII.12  
The student will compute and distinguish between permutations and combinations and use technology for applications.

<table>
<thead>
<tr>
<th>ESSENTIAL UNDERSTANDINGS</th>
<th>ESSENTIAL KNOWLEDGE AND SKILLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>• The <em>Fundamental Counting Principle</em> states that if one decision can be made ( n ) ways and another can be made ( m ) ways, then the two decisions can be made ( nm ) ways.</td>
<td>The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to</td>
</tr>
<tr>
<td>• <em>Permutations</em> are used to calculate the number of possible arrangements of objects.</td>
<td>• Compare and contrast permutations and combinations.</td>
</tr>
<tr>
<td>• <em>Combinations</em> are used to calculate the number of possible selections of objects without regard to the order selected.</td>
<td>• Calculate the number of permutations of ( n ) objects taken ( r ) at a time.</td>
</tr>
<tr>
<td></td>
<td>• Calculate the number of combinations of ( n ) objects taken ( r ) at a time.</td>
</tr>
<tr>
<td></td>
<td>• Use permutations and combinations as counting techniques to solve real-world problems.</td>
</tr>
</tbody>
</table>