Harmonic motion describes the motion of a point on an object that vibrates, oscillates, rotates, or is moved by wave motion. A traditional example is a mass attached to the end of a spring, which bobs up and down. The mass on the spring moves with harmonic motion. Other examples are such things as the vibrations of a guitar string, the pistons of an engine or a pendulum swinging back and forth.

As it turns out, trigonometric functions are very useful for describing this type of motion. In fact, harmonic motion can be represented very well by using a sine or cosine function.

**Definition:** Simple Harmonic Motion

A point that moves on a coordinate line is in simple harmonic motion if its distance ‘\(d\)’ from the origin at time ‘\(t\)’ is given by:

\[
d = A \cos (kt + c) \quad \text{or} \quad d = A \sin (kt + c).
\]

Note:

- amplitude = \(|A|\)
- period = \(\frac{2\pi}{k}\)
- phase shift = \(-\frac{c}{k}\)
- frequency = \(\frac{k}{2\pi}\).

Remember:

The **amplitude** of an object is its maximum displacement from the zero point.

The **period** is the time it takes to complete one cycle.

The **frequency** is the number of cycles per unit of time. The frequency is equal to the reciprocal of the period.
Example 1: A buoy marking a channel in the harbor bobs up and down as the waves move past. Suppose the buoy moves a total of 6 feet from its high point to its low point and returns to its high point every 10 seconds. Assuming that at \( t = 0 \) the buoy is at its high point and the middle height of the buoy’s path is \( d = 0 \), write an equation to describe its motion.

Solution: We will use the formula: \( d = a \cos (kt + c) \) rather than \( d = a \sin (kt + c) \) because the buoy is at its high point when \( t = 0 \) and the cosine parent graph starts at the high point (the phase shift will be 0).

We know that the amplitude \( a = 3 \) (the total vertical distance is 6 ft, so the amplitude is half that) and the period = 10 seconds. Plug this into the formula for period and solve for \( k \):

\[
10 = \frac{2\pi}{k} \quad 10k = 2\pi \quad k = \frac{2\pi}{10} \quad \text{or} \quad \frac{\pi}{5}
\]

Fill in the values for \( a \) and \( k \) and we have:

\[
d = 3 \cos \left( \frac{\pi}{5} t + c \right)
\]

To find \( c \) you can plug in a known ordered pair, i.e., when \( t = 0 \), \( d = 3 \) (the high point).

\[
3 = 3 \cos \left( \frac{\pi}{5} \cdot 0 + c \right)
\]

\[
3 = 3 \cos (c)
\]

\[
\frac{3}{3} = \frac{3 \cos (c)}{3}
\]

\[
1 = \cos (c)
\]

\[
c = \cos^{-1} 1 = 0
\]

The method above will always work for finding \( c \), but we already knew that \( c = 0 \) because the motion started with the buoy at the high point which is where a cosine curve with no phase shift starts.

Our equation is: \( d = 3 \cos \left( \frac{\pi}{5} t \right) \).
Example 2: A weight on a spring bounces a maximum of 8 inches above and below its equilibrium (zero) point. The time for one complete cycle is 2 seconds. Write an equation to describe the motion of this weight, assume the weight is at equilibrium when \( t = 0 \).

**Solution:** We will use the sine function since it starts at equilibrium with 0 phase shift. The formula is \( d = a \sin(kt + c) \).

We know that \( a = 8 \). The period is 2 seconds, \( \frac{2\pi}{k} = 2 \) so \( k = \pi \).

The phase shift is 0, so \( c = 0 \).

Our equation is: \( d = 8 \sin(\pi t) \).

Example 3: Given the equation, \( d = 6 \cos\left(\frac{3\pi t}{4}\right) \), find the following:

a) the maximum displacement from zero,
b) the frequency,
c) the value of \( d \) when \( t = 8 \).

**Solution:** Our general formula is \( d = a \cos(kt + c) \).

a) Displacement from zero is amplitude which equals 6.

b) Frequency is the reciprocal of the period and the period here is

\[
\frac{2\pi}{\frac{3\pi}{4}} = \frac{2\pi}{1} \cdot \frac{4}{3\pi} = \frac{8}{3}
\]

So the frequency is \( \frac{3}{8} \) cycle per unit of time.

c) When \( t = 8 \),

\[
d = 6 \cos\left(\frac{3\pi \cdot 8}{4}\right) = 6 \cos 6\pi = 6(1) = 6.
\]

Example 4: Given the equation, \( d = 5 \sin\left(2t - \frac{\pi}{6}\right) \), find the following:

a) amplitude
b) period
c) frequency
d) phase shift

**Solution:** Our general formula is \( d = a \sin(kt + c) \).

a) Amplitude = 5.

b) The period \( = \frac{2\pi}{2} = \pi \).

c) The frequency \( = \frac{1}{\pi} \) cycle per unit of time.

d) The phase shift \( = -\frac{\frac{\pi}{6}}{k} = -\frac{\frac{\pi}{6}}{2} = \frac{\pi}{6} \cdot \frac{1}{2} = \frac{\pi}{12} \) to the right.
Example 5: Write an equation to represent simple harmonic motion when the initial position is -10, the amplitude is 10 and the period is $\frac{1}{2}$.

Solution: We’ll use the general formula $d = a \cos (kt + c)$. Since the object starts at its low point, phase shift can equal 0.

Amplitude, $a = 10$
Since the period is $\frac{1}{2}$, $\frac{2\pi}{k} = \frac{1}{2}$, $k = 4\pi$,
Phase shift, $c = 0$.

The equation is: $d = 10 \cos (4\pi t)$.

Try these:

a) The motion of a floating leaf can be described by the equation $y = 5\cos \left(\frac{3\pi}{4} t\right)$, where ‘$y$’ is measured in cm and ‘$t$’ is measured in seconds. Find the maximum displacement and the frequency of the leaf.

b) Given $d = 4 \cos 8\pi t$, find the maximum displacement in cm, the period in seconds and the frequency.

c) Given $y = 3 \cos \left(\frac{\pi}{6} t + \frac{\pi}{3}\right)$, find the amplitude, period, phase shift and frequency.

d) A point at the end of a tuning fork moves in simple harmonic motion, described by $d = a \sin kt$. Find ‘$k$’ given that the tuning fork for middle C had a frequency of 264 vibrations per second.