Trigonometry
Solving Triangles, The Law of Sines: The Ambiguous Case

For two triangles to be congruent, all three pairs of sides must be congruent and all three pairs of angles must be congruent. It is sufficient to prove that two triangles are congruent if you are given any of these combinations of congruent parts (this should look familiar from Geometry class!):

- SSS (given that all three pairs of sides are congruent, the angles must be congruent)
- SAS (given that two pairs of sides are congruent and the angles between those sides are also congruent, then the other pairs of sides and angles must also be congruent)
- ASA (given that two pairs of angles are congruent and the sides between those angles are also congruent, then the other pairs of sides and angles must also be congruent)
- AAS (given that two pairs of angles are congruent and that one pair of sides not between those angles are congruent, then the other pairs of sides and angles must also be congruent)

Why is it NOT sufficient to prove that triangles are congruent if they share the same SSA?

We will build a device that shows us how two triangles with the same SSA might *not* be congruent in all aspects.

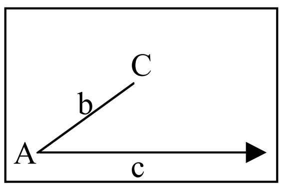
<u>Materials needed for each student</u>:
- a ruler
- scissors
- three different colored markers
- two blank sheets of paper - try to use a heavy bond, card stock if possible
- two strips of card stock, about 10 inches long, half an inch wide
- two paper fasteners (the ones with two branches that fold out to secure papers together)

ACTIVITY 1: Given Angle A, acute

We are going to build a triangle where only three measurements are known: the measure of angle A, and the measures of sides *a* and *b*. (SSA)

Draw the given Angle (A) and one side (b)

1. Students need to use the ruler to draw a *ray* (not a segment) along the bottom (the long edge) of the paper, about half an inch from the edge. Then draw a line segment from the vertex of the ray, forming an ACUTE angle (this is the angle A from SSA). Label the endpoints and sides at shown:

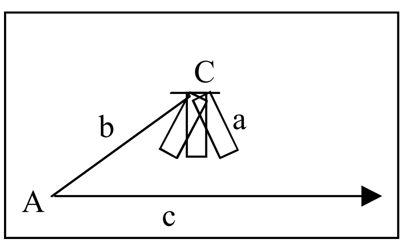


Note that the endpoint, B, is somewhere along the bottom ray. The length of side, c, is unknown, so don't label B yet.

So far, we have fixed two measurements of our triangle, the angle, A, and the length of side *b*. Side *a* would connect angles C and B, but since we don't know the size of angle C or B we don't know how to draw in side *a*.

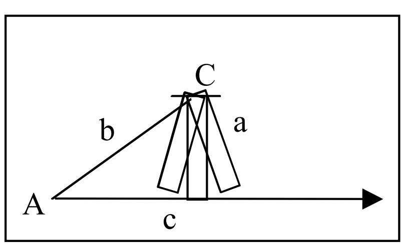
2. Cut a small slit at point C, just wide enough for the strip of card stock to go through. Slide one of the paper strips through the slit. **The length of strip that we pull through the slot is side *a*.** We can pivot the strip, so we aren't assuming the size of angle B or angle C.

3. CASE 1: SIDE '***a***' IS VERY SHORT. Push most of the strip of paper behind the triangle, as if the measure of side ***a*** is very small. Gently pivot the exposed length left and right. What do you notice?



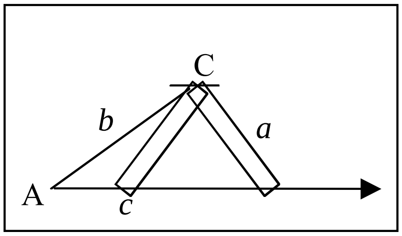
No triangle is formed.

4. CASE 2: SIDE '***a***' IS AN EXACT MEASURE. Pull out just enough of the strip so that it barely touches side ***c*** at only one point, along a line perpendicular to side *c*.



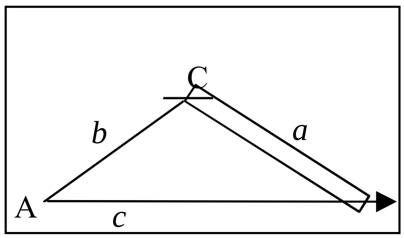
Only one triangle can be formed.

5. CASE 3: SIDE '***a***' IS A LITTLE BIT LONGER. Pull out the strip a little more, as if the measure of side ***a*** were longer that in the last step, but not as long as side ***b***.



*Two triangles can be formed if side **a** is this measurement.*

6. CASE 4: SIDE '***a***' IS MUCH LONGER. Pull out the strip so that the measure of side ***a*** is longer than side ***b***.



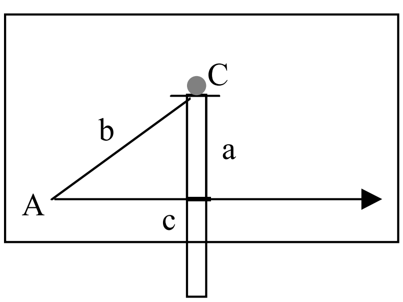
Only one triangle can be formed if side a is this measurement, pivoting the strip to the left puts the triangle on the wrong side of angle A which was known.

FACT: Being given the measures of Angle A, Side *a* and Side *b* (SSA) does not mean we know the measures of the other sides and angles. Depending on the length of side *a*, there may be 0 , 1 or 2 triangles possible with the given measurements.

<u>*Finding the Boundaries of Side '**a**'*</u>

7. Pull the strip out a little longer than in the last step. Push the paper fastener through the paper and the part of the strip still behind the paper at point C. Secure the fastener.

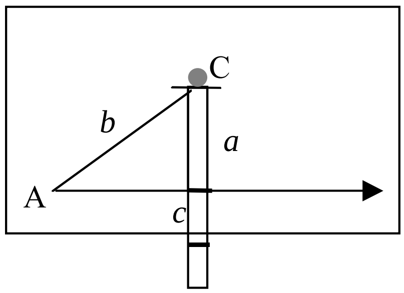
   Lay the strip so that it forms a perpendicular with line ***c***. Draw a line across the strip at this length. This is the length of ***a*** that forms exactly one right triangle. Can you figure out this length based on angle A and side ***b***?



If ***a*** = ***b****sinA, it forms one right triangle.

Use small print to write a=bsinA on this mark.

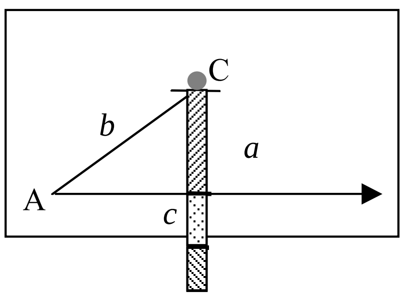
8. Gently pivot the strip so that it is the same length as side ***b***. Draw a line across the strip at this length and pivot the strip back to its normal position.



If *a*= *b*, it forms one triangle, only when line a is pivoted to the right of angle C.

Use small print to write a=b on this mark.

9. Now, if side ***a*** is less than, between, or greater than these marks, we will have 0, 1 or 2 triangles possible. Use your different colored markers to color the exposed parts of the strip:



Make a legend on the side of the paper:
top color:
a < bsinA (0 triangles)
middle color:
bsinA < a < b (2 triangles)
bottom color:
a > b (1 triangle)

Summary:

Condition	**No. of Triangle Solutions**
$a < b\sin A$	0
$a = b\sin A$	1
$b\sin A < a < b$	2
$a \geq b$	1