Trigonometry
Solving Triangles: The Law of Cosines

The Law of Cosines is useful when you are given information about all three sides of the triangle or information about two sides and the included angle. Basically, to use the Law of Cosines you need information about all three sides or two sides and an included angle.

**Law of Cosines:**

Let $\triangle ABC$ be any triangle with $a$, $b$, and $c$ representing the measures of sides that are opposite angles with measures $A$, $B$, and $C$ respectively. Then the following are true:

\[
\begin{align*}
    a^2 &= b^2 + c^2 - 2bc \cos A \\
    b^2 &= a^2 + c^2 - 2ac \cos B \\
    c^2 &= a^2 + b^2 - 2ab \cos C
\end{align*}
\]

But there is one thing that you need to remember about triangles: the sum of any two sides must be greater than the third side. So if you are given all three sides of a triangle, add the two smallest sides together first to make sure that those sides can actually form a triangle. Sometimes the problem tries to trick you.

**Example 1:** Solve $\triangle ABC$ if $a = 67$, $b = 54.1$ and $c = 94.9$. We are given the three side lengths; we need to find the measures of all three angles. Check to see if the measures work for a triangle:

\[
\begin{align*}
    67 + 54.1 &> 94.9 \\
    67 + 94.9 &> 54.1 \\
    54.1 + 94.9 &> 67
\end{align*}
\]

We should start by finding the largest angle first, let’s find angle $C$:

\[
\begin{align*}
    94.9^2 &= 67^2 + 54.1^2 - 2 \cdot 67 \cdot 54.1 \cdot \cos C \\
    9006.01 &= 4489 - 7249.4 \cos C \\
    1590.2 &= -7249.4 \cos C \\
    \frac{1590.2}{-7249.4} &= \cos C \\
    \cos^{-1}\left(\frac{1590.2}{-7249.4}\right) &= C \\
    102.7^\circ &= C
\end{align*}
\]

**Be sure that you do NOT subtract 7415.81 and 7249.4! Remember the order of operations!**
Now we can use the Law of Sines to find the next angle, or you can continue using the Law of Cosines.

\[
\frac{\sin 102.7^\circ}{94.9} = \frac{\sin A}{67}
\]

\[
94.9 \sin A = 67 \sin 102.7^\circ
\]

\[
\sin A = \frac{67 \sin 102.7^\circ}{94.9}
\]

\[
A = \sin^{-1} \left( \frac{67 \sin 102.7^\circ}{94.9} \right)
\]

\[
A = 43.5^\circ
\]

To find the last angle, subtract the first two from 180 degrees.

\[
B = 180 - 102.7 - 43.5
\]

\[
B = 33.8^\circ
\]

We should double check to make sure the results make sense. The longest side is opposite the biggest angle and the shortest side is opposite the smallest angle.

**Important to Note:** Notice that in the first step, the inverse cosine of a negative number is an obtuse angle. Remember that the cosine of an angle in Quadrant 2 or 3 is negative, so the *inverse* cosine of a negative number is an angle in either Quadrant 2 or 3, and in a triangle it has to be in Quadrant 2 to be less than 180. Then we use the law of sines to find the second angle; since the sine of an angle in Quadrant 1 or 2 is always positive, we will always be taking the inverse sine of a positive number. When we take the inverse sine of a positive number using a calculator, the calculator will always assume it is in Quadrant 1 (an acute angle), even if it is really in Quadrant 2. If we had started with a different angle in the first step, the law of cosines would have given us the correct acute angle, but if we tried to use the law of sines in step 2 to find angle C we would have gotten an incorrect acute angle. This is why we start with the largest angle in step 1, because if it is an obtuse angle, the law of cosines will correctly identify it. You do have a choice, though, if you prefer you can use the law of cosines to find the first two angles, then you know the angle will be correctly identified in both cases.
**Example 2:** Solve \( \triangle ABC \) if \( a = 170.5 \), \( b = 52 \), and \( c = 72 \).

\[
52 + 72 < 170.5
\]

No triangle can be formed.

The law of cosines can also be used when you are given two sides of the triangle and the angle between them.

**Example 3:** Solve \( \triangle ABC \) if \( b = 102 \), \( c = 64.5 \), and \( A = 111^\circ \)

We must find side \( a \) first:

\[
\begin{align*}
a^2 &= 102^2 + 64.5^2 - 2 \cdot 102 \cdot 64.5 \cdot \cos 111 \\
a^2 &= 14564.25 - (-4715.40548) \\
a^2 &= 19279.6548 \\
a &= 138.9
\end{align*}
\]

Now use the law of cosines to find another angle, we will find angle \( B \):

\[
\begin{align*}
102^2 &= 138.9^2 + 64.5^2 - 2 \cdot 138.9 \cdot 64.5 \cdot \cos B \\
10404 &= 23439.905 - 17911.805 \cos B \\
-13035.905 &= -17911.805 \cos B \\
\frac{-13035.905}{-17911.805} &= \cos B \\
43.3^\circ &= B
\end{align*}
\]

And the last angle is:

\[
C = 180 - 111 - 43.3 = 25.7^\circ
\]

The largest side (a) was across from the largest angle (A) and the smallest side (c) was across from the smallest angle (C), so we can be confident in our solution.