1987 AB5

The trough is 5 ft long and its vertical cross sections are inverted isosceles triangles with base 2 ft and height 3 ft. Water is being siphoned out of the trough at the rate of 2 cubic ft per minute. At any time $t$, let $h$ be the depth and $V$ be the volume of water in the trough.

(a) Find the volume of water in the trough when it is full.

(b) What is the rate of change in $h$ at the instant when the trough is $\frac{1}{4}$ full by volume?

(c) What is the rate of change in the area of the surface of the water at the instant when the trough is $\frac{1}{4}$ full by volume?

$$V = \frac{1}{2} \cdot 2 \cdot h \cdot 5 = 15 \cdot \frac{f^3}{3}$$

$$(\text{full trough})$$

$$b = \frac{2}{3}$$

$$V = \frac{15}{4} = \frac{1}{2} \left( \frac{2}{3} h \right) \cdot h \cdot 5$$

$$\frac{15}{4} = \frac{5}{3} h^2$$

$$h^2 = \frac{15}{4} \cdot \frac{3}{5} = \frac{9}{4}$$

$$h = \frac{3}{2}$$

$$\frac{dV}{dt} = \frac{10}{3} \left( \frac{3}{2} \right) \frac{dh}{dt}$$

$$-2 = \frac{5}{3} \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{2}{5} \text{ ft/min}$$

(The height is decreasing at a rate of $\frac{2}{5}$ ft/min at that moment)

(c) $SA = 5bV$

$$SA = 5 \cdot \frac{2}{3} h$$

$$SA = \frac{10}{3} h$$

$$\frac{dSA}{dt} = \frac{10}{3} \frac{dh}{dt} = \frac{10}{3} \left( -\frac{2}{5} \right) = -\frac{4}{3}$$

(The surface area is decreasing at a rate of $\frac{4}{3}$ ft$^2$/min at that moment)

1990 AB4

The radius of a sphere is increasing at a constant rate of 0.04 cm/sec.

(a) At the time when the radius of the sphere is 10 cm, what is the rate of increase of the volume?

(b) At the time when the volume of the sphere is $36\pi$ cubic centimeters, what is the rate of change in the area of a cross-section through the center of the sphere?

(c) At the time when the volume and the radius of the sphere are increasing at the same rate, what is the radius?

$$V_{\text{sphere}} = \frac{4}{3} \pi r^3$$

$$\frac{dr}{dt} = 0.04 \text{ cm/sec}$$

(a) when $r = 10$ cm, what is $\frac{dV}{dt}$?

$$\frac{dV}{dt} = \frac{4}{3} \pi \cdot 3r^2 \frac{dr}{dt} = 4 \pi (10)^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 400 \pi \left( \frac{4}{10} \right) = 16 \pi \text{ cm}^3/\text{sec}$$

The volume is increasing at a rate of $16 \pi$ cm$^3$/sec.

(b) $V = \frac{4}{3} \pi r^3 = 36 \pi$

$$\frac{4}{3} r^3 = 36$$

$$r^3 = \frac{36 \cdot 3}{4} = 27 \Rightarrow r = 3$$

cross section is Circle $A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi \cdot 3 \cdot (0.04)$$

$$\frac{dA}{dt} = 0.24 \pi \text{ cm}^2/\text{sec}$$

(c) $\frac{dV}{dt} = \frac{dr}{dt}$

$$\frac{dV}{dt} = \frac{4}{3} \pi \cdot 3r^2 \frac{dr}{dt}$$

$$1 = 4 \pi r^2$$

$$r = \frac{1}{2\sqrt{\pi}} \text{ cm}.$$
1994 AB5

A circle is inscribed in a square. The circumference of the circle is increasing at a constant rate of 6 in/sec. As the circle expands, the square expands to maintain the condition of tangency.

(a) Find the rate at which the perimeter of the square is increasing.

(b) At the instant when the area of the circle is \(25\pi\) sq. in., find the rate of increase in the area enclosed between the circle and the square.

\[
\frac{dC}{dt} = 2\pi \frac{dr}{dt} = 6 \text{ in/sec}
\]

\[
\frac{dr}{dt} = \frac{6}{2\pi} = \frac{3}{\pi}
\]

\[
A = \pi r^2 = 25\pi \Rightarrow r = 5
\]

\[
A_{sq} = (2r)^2 = 4r^2
\]

\[
\frac{dA_{sq}}{dt} = 8r \frac{dr}{dt} = 8(5) \cdot \frac{3}{\pi} = \frac{120}{\pi}
\]

\[
A_{circle} = \pi r^2
\]

\[
\frac{dA_{circle}}{dt} = 2\pi r \frac{dr}{dt} = 2\pi (5) \cdot \frac{3}{\pi} = 30
\]

So rate of increase in area enclosed between circle & sq. = \left(\frac{120}{\pi} - 30\right) \text{ in}^2 / \text{sec}

1995 AB5

Water is draining from a conical tank with height 12 ft and diameter 8 ft into a cylindrical tank that has a base area of \(400\pi\) square feet. The depth \(h\), in feet, of the water in the conical tank is changing at the rate of \((h+12)\) feet per minute.

(a) Write an expression for the volume of water in the conical tank as a function of \(h\).

(c) Let \(y\) be the depth, in feet, of the water in the cylindrical tank. At what rate is \(y\) changing when \(h = 3\)?

\[
\text{cone } \frac{r}{h} = \frac{4}{12} = \frac{1}{3} \quad r = \frac{h}{3}
\]

\[
V_{cone} = \frac{1}{3} \pi r^2 h = \frac{\pi}{3} \left(\frac{h}{3}\right)^2 \cdot h
\]

\[
\frac{dh}{dt} = (h-12) \cdot \frac{f+1}{\text{min}}
\]

\[
V_{cone} = \frac{\pi}{27} h^3 \frac{f^3}{\text{min}}
\]

(b) \[
\frac{dV_{cone}}{dt} = \frac{\pi}{27} \cdot 3h^2 \frac{dh}{dt} \text{ at } h = 3
\]

\[
= \frac{\pi}{9} \cdot 3^2 (h-12) \text{ at } h = 3
\]

\[
= -9\pi \frac{f^3}{\text{min}}
\]

(c) \[
V_{cyl} = \pi r^2 \cdot y = 400\pi y
\]

\[
\frac{dV_{cyl}}{dt} = 400\pi \frac{dy}{dt} \text{ at } h = 3
\]

\[
9\pi = 400\pi \frac{dy}{dt}
\]

\[
\frac{dy}{dt} = \frac{9}{400} \frac{f}{\text{min}}
\]