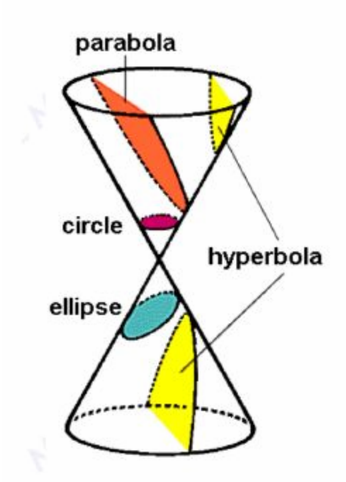
Conics - Ellipses

Conics – General Information

Conics, or conic sections, are plane figures that are formed when you intersect a double-napped cone and a plane. The following diagrams show the different conics that can be formed by a double-napped cone being cut by a plane:



Ellipses: Notice that to create an ellipse by intersecting a cone and a plane, the plane must be askew to the base of the cone, only passing through one of the cones.

Any of these conics can be graphed on a coordinate plane. The graph of any conic on an *x-y* coordinate plane can be described by an equation in the form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

This equation is the *General Form* for all conics. You will see as we study the different conics, their equations and graphs, that we can determine the type of conic given the equation based on the values of the coefficients, A, B, C, D, E and F.

Ellipses:

Looking at the General Form for conics, for an equation in this form to graph an *Ellipse*,

- A and C, the coefficients of x^2 and y^2, must be the same sign, but different numbers.
- B, the coefficient of xy, must equal zero.
- D, E and F can equal any real values.

These equations would all form ellipses on the x-y coordinate plane:

$x^2 + 9y^2 - 2x + 36y + 28 = 0$

$25x^2 + 4y^2 - 50x - 75 = 0$

$9x^2 + 4y^2 - 36 = 0$

To graph an ellipse given the equation in general form, we should first rewrite the equation into the Standard Form for Ellipses:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \quad \text{or} \quad \frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

where (h, k) is the center of the ellipse, a = half the length of the ellipse on the *major axis*, and b = half the length of the ellipse on the *minor axis*. The larger denominator is always a^2. If the larger denominator is under 'x', then the major axis is horizontal, otherwise it is vertical.

Converting an equation from General Form to Standard Form:

1) $x^2+9y^2-2x+36y+28=0$

$x^2-2x+9y^2+36y=-28$

$\left(x^2-2x\right)+9\left(y^2+4y\right)=-28$

Before Completing the Square, for each variable pair, factor out the coefficient of the squared term.

$\left(x^2-2x+\_?\_\right)+9\left(y^2+4y+\_?\_\right)=-28$

Complete the Square.

$\left(\frac{-2}{2}\right)^2=1 \qquad \left(\frac{4}{2}\right)^2=4$

$\left(x^2-2x+\underline{1}\right)+9\left(y^2+4y+\underline{4}\right)=-28+\underline{1\cdot 1}+\underline{9\cdot 4}$

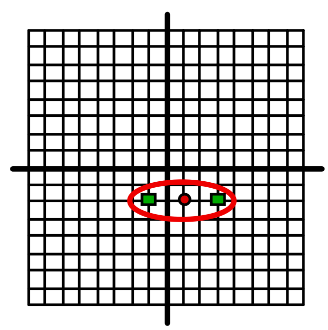
When you add the squares to the right side, multiply each by the factor in front.

$(x-1)^2+9(y+2)^2=9$

$\frac{(x-1)^2}{9}+\frac{9(y+2)^2}{9}=\frac{9}{9}$

Divide both sides by the new constant on right side. This leaves a 1 on the right side.

$\frac{(x-1)^2}{9}+\frac{(y+2)^2}{1}=1$



From the Standard Form, determine the:

Center: (h, k) = (1, -2)

The major axis is horizontal, $a^2=9$

Half the Horizontal Length: $a=\sqrt{9}=3$

Half the Vertical Length: $b=\sqrt{1}=1$

The *vertices* of the ellipse are:

(-2, -2) , (4, -2)

(1, -1) , (1, -3)

For Ellipses, we must also determine the location of the *Foci*. The foci are two points, located along the major axis (here the horizontal axis), inside the ellipse, on either side of the center. The foci are 'c' units away from the center, where $c=\sqrt{a^2-b^2}$.

In this example, $c=\sqrt{9-1}=\sqrt{8}\approx 2.8$. In the graph the foci have been marked with boxes.

The *Foci* of the ellipse are:

$\left(1-\sqrt{8},\,-2\right)$ and $\left(1+\sqrt{8},\,-2\right)$

2) $25x^2 + 4y^2 - 50x - 75 = 0$

$25x^2 - 50x + 4y^2 = 75$

$25\left(x^2 - 2x\right) + 4y^2 = 75$

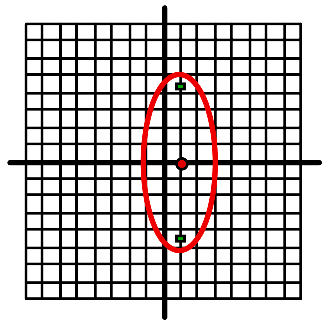
$25\left(x^2 - 2x + \_?\_\right) + 4\left(y^2\right) = 75$

$25\left(x^2 - 2x + \underline{1}\right) + 4\left(y^2\right) = 75 + \underline{25 \cdot 1}$

$25\left(x-1\right)^2 + 4\left(y-0\right)^2 = 100$

$$\frac{25\left(x-1\right)^2}{100} + \frac{4\left(y-0\right)^2}{100} = \frac{100}{100}$$

$$\frac{\left(x-1\right)^2}{4} + \frac{\left(y-0\right)^2}{25} = 1$$



From the Standard Form, determine the:

Center: $(h, k) = (1, 0)$

The major axis is vertical, $a^2 = 25$

Half the Vertical Length: $a = \sqrt{25} = 5$

Half the Horizontal Length: $b = \sqrt{4} = 2$

The *vertices* of the ellipse are:

(-1, 0) , (3, 0)

(1, 5) , (1, -5)

Determine the focal distance:

$c = \sqrt{25 - 4} = \sqrt{21} \approx 4.6$

The foci of the ellipse are on the vertical axis:

$\left(1, \sqrt{21}\right)$ and $\left(1, -\sqrt{21}\right)$