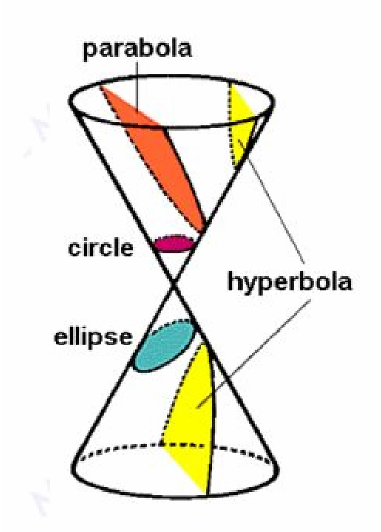
Advanced Algebra
Conics: Hyperbolas

Conics – General Information
Conics, or conic sections, are plane figures that are formed when you intersect a double-napped cone and a plane. The following diagrams show the different conics that can be formed by a double-napped cone being cut by a plane:



Hyperbola: Notice that to create a hyperbola by intersecting a cone and a plane, the plane must intersect at the base so that it cuts through both of the cones.

Any of these conics can be graphed on a coordinate plane. The graph of any conic on an *x-y* coordinate plane can be described by an equation in the form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

This equation is the *General Form* for all conics. You will see as we study the different conics, their equations and graphs, that we can determine the type of conic given the equation based on the values of the coefficients, A, B, C, D, E and F.

Hyperbola:

Looking at the General Form for conics, for an equation in this form to graph a *Hyperbola*,

- A and C, the coefficients of x^2 and y^2, must be opposite signs.
- B, the coefficient of *xy*, must equal zero.
- D, E and F can equal any real values.

These equations would all form hyperbola on the x-y coordinate plane:

$9x^2 - y^2 - 36x - 6y + 18 = 0$
$9x^2 - 16y^2 + 144 = 0$
$9x^2 - y^2 - 36x = 0$

To graph a hyperbola given the equation in general form, we should first rewrite the equation into the Standard Form for Hyperbola:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \text{or} \quad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

where (h, k) is the center of the hyperbola. The denominator of the positive term is a^2, $\boldsymbol{a}$ = distance from the center to the vertex of the hyperbola on the *transverse axis*, and $\boldsymbol{b}$ = distance from the center to the side of the hyperbola on the *conjugate axis*. The lobes of the hyperbola have their vertices on the transverse axis.

Converting an equation from General Form to Standard Form:

1) $9x^2 - y^2 - 36x - 6y + 18 = 0$

$9x^2 - 36x - y^2 - 6y = -18$

$9\left(x^2 - 4x\right) - \left(y^2 + 6y\right) = -18$

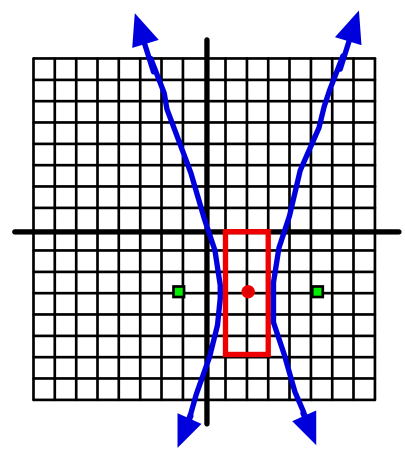
$9\left(x^2 - 4x + \underline{4}\right) - \left(y^2 + 6y + \underline{9}\right) = -18 + \underline{9 \cdot 4} + \underline{-1 \cdot 9}$

$9(x-2)^2 - (y+3)^2 = 9$

$\frac{9(x-2)^2}{9} - \frac{(y+3)^2}{9} = \frac{9}{9}$

$\frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1$

Factor the squared term coefficient out of each variable set – note that we factored -1 out of the y-terms.



From the Standard Form, determine the:

Center: (h, k) = (2, -3)

Since the x-term is positive, the transverse axis is horizontal, $a^2 = 1$.

Half the *Transverse Axis*: $a = \sqrt{1} = 1$

Half the *Conjugate Axis*: $b = \sqrt{9} = 3$

Vertices of the hyperbola are only on the transverse axis:

(1, -3) and (3, -3)

Hyperbola also have *Asymptotes* that determine the curve. To draw the asymptotes, first make a rectangle on the graph, centered at the center of the hyperbola. The length and width of the rectangle equal the transverse and conjugate axes. Then draw the *Asymptotes* as the diagonals of the rectangle, but extend them beyond the rectangle.

Find the slopes of the *Asymptotes*: slopes = ± 3.

The *Lobes* of the hyperbola are drawn from the vertices, (1, -3) and (3, -3), outward, approaching the asymptotes.

The *Foci* are on the transverse axis, '*c*' units from the center, $c = \sqrt{a^2 + b^2}$. Here, $c = \sqrt{1 + 9} = \sqrt{10} \approx 3.2$. The *Foci* are:

$\left(2 - \sqrt{10}, -3\right)$ and $\left(2 + \sqrt{10}, -3\right)$.

2) $9x^2 - 16y^2 + 144 = 0$

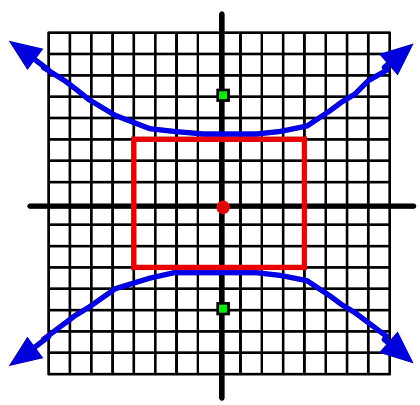
$9x^2 - 16y^2 = -144$

$\frac{9x^2}{-144} - \frac{16y^2}{-144} = \frac{-144}{-144}$

$-\frac{x^2}{16} + \frac{y^2}{9} = 1$

$\frac{y^2}{9} - \frac{x^2}{16} = 1$

Since there are no linear terms we do not need to complete the square.



From the Standard Form, determine the:

Center: $(h, k) = (0, 0)$

Since the y-term is positive, the transverse axis is vertical, $a^2 = 9$.

Half the *Transverse Axis*: $a = \sqrt{9} = 3$

Half the *Conjugate Axis*: $b = \sqrt{16} = 4$

Vertices of the hyperbola are only on the transverse axis:

(0, 3) and (0, -3).

The slopes of the *Asymptotes* are:

slopes = $\pm\frac{3}{4}$

The *Lobes* of the hyperbola are drawn from the vertices (0, 3) and (0, -3), outward, approaching the asymptotes.

The *Foci* are on the transverse axis, '*c*' units from the center, $c = \sqrt{a^2 + b^2}$. Here, $c = \sqrt{9 + 16} = \sqrt{25} = 5$. The *Foci* are:

(0, 5) and (0, -5).